Circuits and Machines in ELECTRICAL ENGINEERING

Volume I. CIRCUITS

JOHN O. KRAEHENBUEHL, M.S., E.E.
Professor of Electrical Engineering, University of Illinois

MAX A. FAUCETT, M.S., E.E.
Professor of Electrical Engineering, University of Illinois

Second Edition
1947

New York · JOHN WILEY & SONS, Inc.
London · CHAPMAN & HALL, Limited

PRINTED IN THE UNITED STATES OF AMERICA

PREFACE TO SECOND EDITION

The philosophy of the second edition of the text does not differ from that of the first edition, and the arrangement of the text remains the same. Since the text has been used for electrical engineering students as well as for other engineering students, the material has been changed to meet both needs. The request for the additional material influenced the decision to divide the text into two volumes, one on circuits and the other on machines.

The first volume is confined to the electrical circuits, and the material emphasizes the circuit as the basic element in the machine and in the study of machine operation, rather than as a part of the communication circuit. The emphasis is upon power.

The chapter on electronics has been expanded to include more information that is applicable to the field of industrial control. The material on ignitrons is new. The original chapter on converters and rectifiers has been eliminated; the discussion of machines has been placed in Volume II, and the remainder has been included in the electronics chapter. Portions of other chapters have been expanded and clarified to add to the usefulness of the text; many of these changes were made upon the request and advice of users of the text.

Since the mks system of units is meeting with considerable favor in the engineering field, the first chapter includes some material on the use of the system, but the text material follows the cgs system because the student enters the course with a physics background and the physicists have not adopted the mks system.

The appendix, on meters, is retained as part of Volume I. New problems have been added throughout the text and, as in the first edition, the attempt has been made to give problems which lead to analysis rather than to substitution into expressions which appear in the developments.

There have been some requests for laboratory experiments which could be adapted to the text material. The authors do not consider these a part of the text material; very satisfactory results have been obtained by using *Circuit Analysis by Laboratory Methods*, by Professor C. E. Skroder and Professor M. S. Helm, in the supplementary laboratory course at the University of Illinois.

The authors are indebted to those who have contributed ideas and suggestions for making the revisions in this new edition.

J. O. K.
M. A. F.

UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS
May 2, 1947

PREFACE TO FIRST EDITION

This book, which is written for those who wish a knowledge of the fundamentals in electrical engineering, may be used as a first course for electrical engineering students, but it is written primarily for non-electrical engineering students who take electrical engineering as a secondary subject. For a period of twelve years the authors have used this method of presenting electricity as a general subject, dealing with the general system and developing the specific system as a special case. By combining the concepts of alternating current and direct current, it is possible to develop the subject with less repetition. The student, moreover, is not handicapped by having first to study principles of direct current and later to modify his point of view to develop the work in the more general field of alternating-current electricity. The rapid expansion of the subject has made it necessary either to neglect that which is new or to eliminate much that is basically necessary. By using the new method of approach, the authors find that the student is less confused and, duplication having been eliminated, there is time for material on new developments.

This book is the revision of a preliminary work which has passed through two editions and three years of classroom use, embodying ideas of four instructors and comments from many interested students. The electrical circuit is developed with the view of using circuit fundamentals in the analysis of machines. In developing the subject, stress is placed on physical analysis rather than upon mathematical derivations. If a mathematical development seems advisable, however, it is included but in such a way as not to encourage memorizing. The material does not develop either new or radically different interpretations but approaches the material from a new method of analysis. Demonstration problems and figures are kept simple, thereby eliminating much descriptive material in the text proper. The subject is treated qualitatively rather than in a quantitative or design manner. Those things that make only one or two per cent difference in the result are neglected or merely mentioned and not elaborated. Basic ideas are frequently repeated and reference is made to fundamental material in previous chapters, not by paragraphs but as to chapter content, so that the student will be encouraged to consider the whole instead of only a part of the correlating material. By means of this repetition, the connecting link between electric and magnetic circuits and their relationships to the machines is fused.

The problems have been chosen for the purpose of developing analysis, not for substituting numerical data into derivations developed in the book. The working of the problems should enable the student to recognize and use the fundamental principles that underlie the functioning of electrical circuits and the operation of machines. Electronics, which occupies approximately five per cent of the text, has not been included for its novelty, consideration being given only to fundamentals and equipment found in use at present. The specialist will find that his field has been neglected, for it is not the intent of the book to be encyclopedic in nature. These special subjects will be mentioned for the student who is interested in doing additional outside work, but they are not handled in detail. The chapters which explain applications are necessarily merely an outline of the possible treatment of this division.

Acknowledgment is made at the point of insertion for the material which has been furnished by manufacturers and commercial organizations. The authors recognize, in special appreciation, the assistance given by Mr. H. N. Hayward and Mr. L. L. Smith, members of the electrical engineering staff at the University of Illinois, who have taught classes in which the preliminary editions have been used. Their criticisms and suggestions have aided the authors to develop approaches and clarify vague statements. Mr. Hayward has contributed one-third of the problems in the book. Professor C. A. Keener read and criticized the preliminary material on machines. Mr. E. F. Heater of the Engineering Experiment Station aided in the checking and arrangement of the drawings.

J. O. K.
M. A. F.

University of Illinois
Urbana, Illinois
November, 1938

CONTENTS

CHAPTER 1

FUNDAMENTAL CONSIDERATIONS IN THE FIELD OF ELECTRICAL ENGINEERING

1. Fundamental Factors. The following should be considered in the study of the subject of electrical engineering:

1. The general nature of electrical engineering knowledge.
2. Professional use of electrical engineering knowledge.
3. Natural laws in general.
    a. Cause.
    b. Opposition or counteraction.
        1. Energy consuming (irreversible).
        2. Non-energy consuming (reversible).
    c. Effect.
4. Analysis of information and problems.
    a. Definition.
    b. Statistical or experimental facts.
    c. Mathematical forms.
    d. Derived forms.
5. Units.
    a. Scientific.
    b. Practical.

2. Viewpoint. Certain principles common to all the different branches of electrical engineering should appear in any textbook dealing with the subject; other principles are confined to various divisions of the subject and should be stressed in the specific field only. Regardless of the branch of engineering or division of any specific branch considered, there are general natural laws which are all-inclusive and deal with the physical phenomena. For every cause there is an effect, but the useful effect is not a full value for the applied cause, because of an opposition which either does or does not consume energy. In addition to the physical aspect (though by far the greater portion of scientific investigation is of the physical), there is analytical development of the subject matter. Much of the material spoken of as analytical research is actually done in the field of physical observations, where the results are obtained by trial

and error in the observation of the effect following cause with the opposition factors controlled.

The units, which are definitional and established for quantitative consideration, should be chosen in such a way as to eliminate unruly numerics which lead to possible errors. The units used by the physicist and the engineer are different, not because one set of units is superior to the other, but because the application to actual physical conditions, in each instance, should give numerics easy to manipulate in computations.

The subject of electrical engineering, the applied science dealing with power electricity, has five important divisions, namely: generation, transmission, distribution, control, and utilization. In the last division the subject enters extensively into the various other branches of engineering. In all the divisions, the circuit, a conducting part or a system of conducting parts through which an electrical current is intended to flow, is an important factor, and the laws of the electrical circuit are the coordinating principles. In the analysis of electrical systems, all machines and equipment are reduced to elementary circuits to facilitate calculations and to simplify presentation. Ability to analyze a circuit is an important part of general engineering skill, for the circuit is present in all branches of engineering and proficiency acquired in one branch may be transferred to another in principle, if not in specific application. If the universal nature of the circuit is kept in mind, transfer of technique from other branches of engineering to electrical engineering will, to some extent, alleviate the groundless apprehension, often felt by those taking survey courses, regarding the difficulty and abstractness of the subject.

To the average individual, electricity is surrounded by mystery. Lightning, as the symbol of all electricity, is associated with many superstitious practices and is considered a powerful, unexplained force. This leaves an early impression of fear and mystery, which later broadening of knowledge does not completely erase. For this reason, electricity has been placed in the realm of the abstract. The constant plea of the beginner is to have electricity materialized. Numerous analogies have been developed which attempt to explain electrical current flow by mass translation or liquid flow, but they are aids of questionable value in the development of a proper attitude toward the study of electricity.

The student has mastered many equally abstract phases of engineering, solving without hesitance problems in mechanics dealing with such properties as inertia and radius of gyration. In the laboratory, a gage reading in pounds per square inch is completely satisfactory to the aver-

age student, who takes for granted that he knows what is in the pipe, whereas any number of substances may give the same pressure reading, because the gage does not indicate the type of fluid in the pipe. However, in the electrical laboratory the voltmeter is rarely accepted as an indicator with equal confidence. There is always a request, either spoken or implied, for materialization of electricity or its associated phenomena.

Experience is a factor that tends to alter the approach to the basic concepts. Other phases of engineering have usually presented their problems early in life, and for years decisions have been made dealing with cause and effect. In electricity, the average individual seldom finds problems that must be solved. Electrical equipment is installed with a view to safety and, therefore, the greater portion of the equipment is shielded with safety devices, precluding handling or observing of the installation. The sooner the individual perceives that electrical, hydraulic, dynamic, and chemical forces are governed by the same general laws, just that soon will the mind be framed to accept and correlate the electrical laws with actual experiences and with other branches of engineering.

3. Purpose and Extent of Subject Matter. A subject must be approached with a clear understanding of the extent to which it is to be covered. There are three types of interest that may be embodied in the study of an engineering subject: *research*, *design*, and *application*, each dealing with a different viewpoint of analyzation. The first, research, is the investigation of some little-understood detail and the analysis of this detail, and is a highly specialized field. In considering the last two divisions (design and application), each branch of engineering makes a detailed study of the known knowledge in the field, with particular attention to quantitative knowledge.

The professional study may take three different paths: one, dealing with some highly specialized branch of the subject; two, a broader knowledge of application and design; three, another group (professions that need information in other professional fields) finds it necessary to have a general knowledge of the subject but is not interested in details of design or special quantitative knowledge. This text material has been arranged for the latter group. These individuals should consider the subject with regard to application, should understand electrical terms, and should be able to pass judgment on electrical problems. This type of knowledge might be classified under the term qualitative knowledge, as contrasted to quantitative knowledge.

4. Laws of Nature. Man's method of passing on knowledge is by the use of definitions. For a time, anyone taking up a new subject must use

rote to a great extent, but the ease with which definitions are changed to *concepts* marks the beginning of understanding of a subject. The easiest way to build up concepts is not only to use such information as may be accumulated with each new assignment but also to associate the new subject with phases of natural phenomena already mastered.

Nature, being consistent, has not created laws for light, heat, and mechanics which differ from those for electricity. Confusion is caused by man's definitions and interpretations of natural laws. In electricity alone, analysis of the actions of nature by two different methods, the

TABLE I–1

PHYSICAL PROPERTIES OF THE ELECTRICAL SYSTEM

Cause	(E) Electromotive Force		
Opposition	(Z) Impedance	Resistance (R)	
		Reactance (X)	Inductive ($\omega\mathcal{L}$)
			Capacitive $\left(\frac{1}{\omega C}\right)$
Effect	(I) Current		

electromagnetic and electrostatic, has been attempted. This gives rise to three distinct systems of units, of which two are fundamentally different, whereas the third exists in order that the first two may be given practical application. With this difference in one branch of science, what could be expected but absolute confusion among the different branches? However, there are some very general laws that are common to all subjects; for instance, to every action there is an equal and opposite reaction.

Nature always balances accounts for, to satisfy the Law of Conservation of Energy, in the cycle all energy is accounted for by various phenomena. These, however, are often so involved, in both character and description, that only a few of the contributing factors are understood. For the present, the problems of the engineer are solved by the analysis of the three functions of natural phenomena: *cause*, *opposition*, and *effect*. Table I–1 shows the division in the electrical system.

5. Cause. In the electrical system the cause is voltage (electromotive force). Electromotive force lies behind the phenomenon that must be explained and compares with the force element considered in mechanics.

It is well to locate this element in any system that is being studied, for, without proper understanding of this fundamental, a problem will seem to be without a solution.

6. Effect. Effect is the end result—the useful energy effect of current never measuring up in magnitude to the energy present in the cause. Engineering has devoted much of its endeavor to obtaining a full measure of effect from every cause. In the electric system, the current is the normal effect observed; in mechanics, mass translation or rotation is the normal effect.

7. Opposition. Opposition might properly be called the counteraction, and, although frequently condemned by the engineer as a hindrance, it is often of use in producing the desired effect. There are two types of counteraction, namely, that which consumes energy and that which stores energy for a period of time and returns it to the system. The first may be called *irreversible* counteraction and the second, *reversible* counteraction. Both types are present in every natural phenomenon but experience is required in determining the weight of each in the resultant effect. Though opposition stands between the engineer and the perfect machine, there are times when it adds the desired amount of stability to the system.

a. Irreversible Counteraction. This involves an energy loss to the electrical system that cannot be regained. This loss may be used to do work, as in the electric water heater and the electric iron where the electrical energy is converted into thermal energy; or in the motor where the conversion is to mechanical energy. That energy which is considered a total loss (such as the heat radiated from wires carrying current) is not used by any other system but goes to the leveling of the energy planes. Resistance in the electrical system and friction in the mechanical system are examples of irreversible counteraction.

b. Reversible Counteraction. This form of opposition does not remove energy from the system; it only stores energy for a period of time, returning an equal amount to the system at the completion of the energy cycle. Examples of this type of counteraction are, in the electrical system, inductance and, in the mechanical system, inertia.

8. Treatment of Engineering Knowledge in Answering Natural Problems. The engineer attempts to solve any problem presented by combining the general and specific laws of nature. Complete knowledge of a subject may be divided into four classes (Table II–1): *definition, statistical facts, mathematics,* and resultant *derived forms* or mathematical expressions. The first two belong to the subject being studied, whereas the third belongs to every branch of science, in fact to all knowledge, the difference being only in application of the principles. The last is

the highest type of development and depends upon a combination of the first three. The derived form furnishes the general formulas of practice, develops formulas in fields yet unknown, and proves phenomena that have been noted without adequate explanation.

TABLE II–1

Analysis of the Engineering Problem

Analytical	Physical
1. Definition	1. Cause
a. Rote	2. Opposition
b. Concept	3. Effect
2. Experimental data	
3. Mathematics	
4. Developed expressions	

9. Definition. The normal approach to any subject is through its definitions. In an attempt to pass on information it becomes evident that a thing must be so described that a mental picture can be formed. How often are words adequate to express the proper picture? Repetition of the words expressing a definition in no way adds to the knowledge of an individual; those words must create some picture that can be interpreted for future use, and the best approach is through utilization of facts already learned.

The common definition of any ordinary thing, such as a dog, calls to the minds of those hearing the word a variety of pictures, from the most specific dog in one person's experience to an abstract animal having the attributes of the general class. To the scientist in the field of zoology there is a series of words definitely locating the dog where it belongs in the general class of mammals. All engineering definitions attempt this specific type of classification, but even then this can be only a basis for the foundation of knowledge and not the knowledge itself.

Replacement of the definition by a concept gained through use or experience should be the object in an analysis and study of the subject. In the concept, words begin to fail in setting forth the picture, for the concept is, paradoxically, both all inclusive and very selective. Everything pertaining to the thought or object is present, but only those points needed in answering the question at hand come to the front. After the definition is memorized, increasing knowledge and additional experience help to broaden the definition to an elementary concept which becomes more comprehensive with further study, but never is complete. Figure 1–1 presents graphically an incomplete concept of electrical resistance (R). This figure is introduced at this time to illustrate the extent of

the development of a concept, by a qualitative approach, and not for specific information. After a study of the text through Chapter 14 the student can check this figure for additional material not here explained.

RESISTANCE

ELECTRICAL — PHYSICAL

Power	Circuit	Parameters	Physical
$P = I^2R$	$E = I\sqrt{R^2 + X^2}$	$Z = \sqrt{R^2 + X^2}$	ρ = material characteristic
$P = \dfrac{E^2R}{R^2 + X^2}$	$\sqrt{R^2 + X^2} = \dfrac{E}{I}$	$g = \dfrac{R}{R^2 + X^2}$	A = cross-sectional dimension
$\cos\theta = \dfrac{R}{\sqrt{R^2 + X^2}}$	$I = \dfrac{E}{\sqrt{R^2 + X^2}}$	$b = \dfrac{X}{R^2 + X^2}$	l = length dimension
$\sin\theta = \dfrac{X}{\sqrt{R^2 + X^2}}$		$R = \dfrac{g}{g^2 + b^2}$	t = temperature
		$R = Z\cos\theta$	
		$X = \sqrt{Z^2 - R^2}$	

FIG. 1-1. A graphical concept of electrical resistance.

10. Statistical Facts. Every subject has basic truths that have been discovered. These are confirmed by many observations, but there is always doubt whether all factors have been properly evaluated and considered. For example, Ohm's Law was sufficient for a time, but other facts were added as general knowledge expanded. Though the law is still satisfactory under certain conditions, it has only limited application. To illustrate, it can readily be seen that, if equipment designs were made where temperature change did not occur, there would never appear in practice a need for considering temperature as a variable. The engineer would not consider this a factor in design and, from his viewpoint, it would be unnecessary. However, a physicist, in pure science, might formulate a theory considering temperature change and its influence; and the engineer, confronted in a problem by temperature change, would utilize the theory of the physicist. This suggests the relative dependence of engineering on the pure science that precedes application.

The question of difference of viewpoint in the fields of engineering and physics frequently comes to the front. Where to pure science zero

and infinity are important considerations, to the engineer they are limits representing either unsatisfactory conditions of operation or destruction. Since electrical engineering is the youngest of the engineering group, here the difference is found most acute; and the purpose of both groups must be considered as well as their interpretation.

11. Mathematics. The general laws of nature are embodied in the subject of mathematics. They are equally applicable to all branches of engineering and, when used in conjunction with a specific form, determined by experimental investigation, lead to the multitude of developed expressions commonly called formulas.

The place of mathematics in the development of the electrical subject may be demonstrated by the following.

From experimental data, it has been determined that:

$$V = IR$$

and

$$R = \rho \frac{l}{A}$$

These factors rest upon many investigations of specific physical and electrical properties. In the first expression V is the voltage, I is the current, and R is resistance. In the second, ρ is the coefficient of resistivity, l is the length, and A is the cross-sectional area of the material under investigation (the area is taken at right angles to the flow of current). By algebraic treatment, the two expressions may be combined into

$$V = I\rho \frac{l}{A}$$

which shows the relationship the applied voltage bears to the dimensions of the system. By using the simple relationship between area (A), length (l), and volume (v), where

$$v = Al$$

a direct substitution in the above form gives:

$$V = I\rho \frac{l^2}{v}$$

or

$$V = I\rho \frac{v}{A^2}$$

where the voltage is in direct proportion to the volume and in inverse proportion to the area squared.

Starting with two simple experimental electrical relationships, it has been possible by mathematics to investigate other possible relationships without recourse to the accumulation of statistical information. These additional forms may be classified as developed expressions or *derived* forms.

12. Derived Forms. The three foregoing forms of information—definitional, statistical, and mathematical—supply the inquisitive mind with the tools for investigation of any subject. By proper application of the general laws to the specific information in the subject, resultant forms will be derived, some of which may be the interpretation of phenomena that exist; others merely predict a possibility. For instance, Maxwell (1873), by an ingenious use of mathematics in the study of electrical phenomena, placed on paper many possibilities not realized in practice until years later.

The normal practice is to memorize these derived forms, or formulas, assuming in so doing that a knowledge of the subject is being gained. More effort should be directed to the first three divisions outlined. If these are mastered, not only do the derived forms present themselves clearly for analysis, but also the mental equipment is such that the problems presented may be approached with the correct tools for scientific investigation. If the mind is supplied with these basic considerations, it ceases to be a filing cabinet and becomes a reasoning machine. Neglect of proper mental approach and the attempt to retain formulas by rote lead rapidly, in the study of electrical engineering, to faulty conclusions, and details soon exclude the capacity to reason.

13. Practical Units. In the study of physics, attention is centered about two systems of units: the electrostatic and the electromagnetic. To the engineer who does not specialize in research or design of electrical circuits and equipment, these two systems are a mass of confused definitions, having neither logical nor physical interpretation. For ordinary use there has been evolved another system of units which is called "practical." This system is not an isolated group of units, but is an *integral power* to the *base ten* of the absolute system. The units of the "practical" system have been accepted by most governments. They are, therefore, international and may be applied or measured in any country without using conversion tables. When the engineer and physicist wish to discuss the field of electricity, the conversion of units from the practical to the absolute system takes some form of ten to an integral power.

The electrical system of units may be transferred to the mechanical system by means of those units of energy which are common to both electrical and mechanical phenomena. The "practical" system holds to the metric units, avoiding much of the confusion which has been

caused in the past by the older system of notation used in other branches of engineering. This book will confine all developments and problems to the use of "practical" units and, when fundamental analysis makes it necessary to convert from the absolute system to the "practical," the value of ten raised to the proper integral power will be introduced.

The "practical" system of electrical units is one in which the units are multiples or submultiples of the cgs electromagnetic system units, given as:

AMPERE	10^{-1} abampere
COULOMB	10^{-1} abcoulomb
VOLT	10^{8} abvolt
OHM	10^{9} abohms
MHO	10^{9} abmho
HENRY	10^{9} abhenry
FARAD	10^{-9} abfarad
JOULE	10^{7} ergs
WATT	10^{7} ergs per second
WATTHOUR	3600 joules

The above units are best defined by the "International System of Electrical and Magnetic Units." *

International Ampere. The international ampere is defined as the current which will deposit silver at the rate of 0.00118800 gram per second (experimental results, one international ampere 0.99985 abampere).

International Coulomb. An international coulomb is the quantity of electricity which passes any section of an electric circuit in one second, when the current in the circuit is one international ampere (0.99985 abcoulomb).

International Ohm. The international ohm is defined as the resistance at zero degrees centigrade of a column of mercury of uniform cross-section, having a length of 106.300 centimeters and a mass of 14.4521 grams (1.00048 abohms).

International Volt. The international volt is the voltage that will produce a current of one international ampere through a resistance of one international ohm (1.00033 abvolts).

International Henry. The international henry is the inductance which produces an electromotive force of one international volt when the current is changing at the rate of one international ampere per second (1.00048 abhenrys).

* A.S.A.—From 1941 "American Standard Definitions of Electrical Terms." Definitions from this source will be used wherever possible in the text.

International Farad. The international farad is the capacitance of a capacitor if a charge of one international coulomb produces a potential difference between the terminals of one international volt (0.99952 abfarad).

International Joule. The international joule is the energy required to transfer one international coulomb between two points having a potential difference of one international volt (1.00018 abjoules).

International Watt. The international watt is the power expended when one international ampere flows between two points having a potential difference of one international volt (1.00018 abwatts).

At the time of their establishment the units of the "practical" system were so selected that the more important units would be of convenient size and that, in the most common electrical equations, the proportionality factor would be unity.

The relationship between the "practical" units can be expressed by equations in which

$$E = IR$$

$$Q = It$$

$$E = -\mathcal{L}\frac{dI}{dt}$$

$$Q = CE$$

$$P = I^2R$$

$$W = EIt$$

E is electromotive force in volts.
I is current in amperes.
R is resistance in ohms.
Q is quantity in coulombs.
$\mathcal{L}$ is inductance in henrys.
C is capacity in farads.
P is power in watts.
W is energy in joules.
t is time in seconds.

14. MKS Electromagnetic System of Units. This system of units is based on the meter, kilogram, and second. It is an expansion of the "practical system" to give values to quantities used as cgs units. It was approved by the International Electrochemical Commission in 1935, except for a difference of opinion on the question of rationalization. The magnetic units established by equations containing 4π are called unrationalized. If the definitions are so chosen as to eliminate

4π from the most frequently used equations the system is said to be one that is rationalized.

The advantage of the mks system of units, when final agreement can be obtained on rationalized quantities, is that in electrical engineering

TABLE III-1

UNITS AND CONVERSION FACTORS

Symbols	Quantity	MKS Units	Conversion Factors (CF) esu	Conversion Factors (CF) emu	CGS Units emu
M	mass	kilogram			gram
l	length	meter			centimeter
t	time	second			second
I	current	ampere	$10^{-1}c$	10^{-1}	abampere
Q	charge	coulomb	$10^{-1}c$	10^{-1}	abcoulomb
V	voltage	volt	$10^{8}c^{-1}$	10^{8}	abvolt
R	resistance	ohms	$10^{9}c^{-2}$	10^{9}	abohm
$\mathcal{L}$	inductance	henry	$10^{9}c^{-2}$	10^{9}	abhenry
C	capacitance	farad	$10^{-9}c^{2}$	10^{-9}	abfarad
F	force	newton	10^{5}	10^{5}	dyne
P	power	watt	10^{7}	10^{7}	$\frac{\text{erg}}{\text{second}}$
W, J	energy	joule	10^{7}	10^{7}	erg
$\mathfrak{F}$	magnetomotive force	pragilbert	$10^{-1}c$	10^{-1}	gilbert
$\mathcal{H}$	magnetic intensity	praoersted	$10^{-3}c$	10^{-3}	oersted
ϕ	magnetic flux	weber	$10^{8}c^{-1}$	10^{8}	maxwell
$\mathcal{R}$	reluctance	$\frac{\text{pragilbert}}{\text{weber}}$	$10^{-9}c^{2}$	10^{-9}	$\frac{\text{gilbert}}{\text{maxwell}}$
$\mathcal{P}$	permeance	$\frac{\text{weber}}{\text{pragilbert}}$	$10^{9}c^{-2}$	10^{9}	$\frac{\text{maxwell}}{\text{gilbert}}$
μ	permeability		$10^{7}c^{-2}$	10^{7}	
k	permittivity		$10^{-11}c^{2}$	10^{-11}	
$\mathcal{B}$	magnetic flux density	$\frac{\text{weber}}{\text{meter}^2}$	$10^{4}c^{-1}$	10^{4}	gauss

$c \approx 3 \times 10^{10}$ cm/sec

CF (conversion factor)

10 amperes $= 10 \times CF$ abamperes
$= 10 \times 10^{-1}$ abampere
$= 1$ abampere

absolute units could be abolished and the practical units, volts, amperes, ohms, watts, joules, would be used in fundamental developments instead of the cgs units and conversion factors. This system is extended to the field of mechanics as well.

Since physicists and chemists, for the most part, oppose the introduction of the mks system, the cgs system will continue in use in most teaching of fundamental concepts. Since the mks units are used by the electrical engineers, in the present "practical units," the mks system will doubtless be extended further in both engineering computations and textbooks. The mks system, if used to replace the three systems of units used at present, would give an ideal solution to the confusion of units. The action of the Commission, however, introduces still another system into the already undesirable state of derangement. This text will use the cgs concept (see Table III–1) in basic discussions because the student will be trained in these by the physics groups; absolute units will be used only when unavoidable.

15. Similarity between Physical Phenomena. Tables IV–1 and V–1 (p. 14) point out how closely electrical engineering follows the other branches of engineering in its analysis of the three major factors—*cause, opposition,* and *effect.*

Since natural laws are expressed by the use of algebraic equations, a comparison of the electrical and magnetic forms with the mechanical subject matter facilitates the transfer of knowledge from one to the other. Table V–1 shows the common algebraic expressions in the electrical, magnetic, and mechanical fields and the general expression utilized in electrical and mechanical engineering. It is not the intention that these be absorbed in their present form but that the whole of the field be viewed as definitely related.

In a closed system:

$$\text{Electrical} \quad e = Ri + \mathcal{L}\frac{di}{dt} + \frac{1}{C}\int i\,dt$$

$$i = \frac{e}{R} + \frac{1}{\mathcal{L}}\int e\,dt + C\frac{de}{dt}$$

R is resistance, $\mathcal{L}$ is inductance, C is capacitance, e is voltage, i is current.

$$\text{Mechanical} \quad f = Kv + M\frac{dv}{dt} + \frac{1}{E}\int v\,dt$$

$$v = \frac{f}{K} + \frac{1}{M}\int f\,dt + E\frac{df}{dt}$$

K is the damping constant (friction effect), M is the mass, E is the resilience constant (elasticity effect), f is the force, and v is the velocity.

Either of these may be written *cause* equals *effect* working through *reversible counteraction* plus *effect* working through *irreversible counteraction.*

TABLE IV-1

QUALITATIVE COMPARISON OF PHYSICAL PHENOMENA IN VARIOUS ENGINEERING FIELDS

Engineering Field	Cause	Opposition or Counteraction	Effect
Electrical	Volts	Resistance Inductance Capacitance	Current
Mechanical	Force	Friction Inertia Elasticity	Velocity
Hydraulic	Head	Friction Inertia Elasticity	Flow
Steam	Pressure	Friction Radiation Conduction Inertia Elasticity	Flow

TABLE V-1

QUANTITATIVE COMPARISON OF PHYSICAL PHENOMENA

Magnetic	Dielectric	Kinetic
$e = N \dfrac{d\phi}{dt} \times 10^{-8}$	$i = \dfrac{dq}{dt}$	$v = \dfrac{ds}{dt}$
$e = \mathcal{L} \dfrac{di}{dt}$	$i = C \dfrac{de}{dt}$	$F = Ma = M \dfrac{dv}{dt}$
$w = p\,dt$	$w = p\,dt$	$w = p\,dt$
$W = \mathcal{L} \int i\,di$	$W = C \int e\,de$	$W = M \int v\,dv$
$W = \frac{1}{2}\mathcal{L}i^2$	$W = \frac{1}{2}Ce^2$	$W = \frac{1}{2}Mv^2$

Any method of analogy presupposes complete knowledge of the subject from which the analogy is drawn; but, since this is frequently lacking, it is often more desirable to develop exact concepts in electrical engineering than to use analogous material from other branches. Comparing electrical current flow to water flow is very well as long as the flow is steady, because experience is rather broad concerning constant flow; but, when electrical current is alternating, the water flow should be reciprocating to carry out the analogy. Since this is rarely considered in elementary courses, to the student, the theory in hydraulics is as vague as in electricity, and the analogy becomes useless.

These tables are not given for the purpose of direct transfer of specific knowledge from the mechanical to the electrical but to point out that the method of approach is the same, merely being centered about a different subject with new definitions and the necessity of building up new concepts. As the subject matter is developed, reference to Tables IV–1 and V–1 will help in assigning the proper positions to electrical phenomena.

PROBLEMS

1–1. Determine the resistance which will draw 10 amp when 100 volts from a source are applied. (*a*) Express in electromagnetic units; (*b*) convert the electromagnetic units into mks units.

2–1. How many abcoulombs will 100 volts deliver to a 5-farad condenser? Change the abcoulombs into mks units.

3–1. A resistance of 10 ohms draws 4 amp from a line. (*a*) Express the power in ergs per second; (*b*) change the answer to mks units.

4–1. If the flux (ϕ) is expressed in webers and the flux density (β) in webers per meter squared, determine the conversion factors by which each must be multiplied to change the mks units into electromagnetic units.

5–1. If 100 volts delivers 4 coulombs of electricity per second, (*a*) what is the energy in joules? (*b*) What absolute current flows? (*c*) Change the units to the mks system.

6–1. If a current of 2 amp flows for 4 sec, (*a*) how many abcoulombs are delivered? (*b*) Determine the absolute units from the mks units.

7–1. If 3 joules of energy are delivered by 10 abamp from a 100-volt source, how long does the current flow?

8–1. A source of 10 volts charges a 40-microfarad condenser in 10 sec. What was the rate of flow of electricity in (*a*) coulombs per second, (*b*) amperes?

9–1. Determine the power in watts when 10^{-7} abvolt are applied to a system of 10^{-8} abohms.

10–1. Determine the ergs per second when 5 coulombs of electricity pass through 10 ohms of resistance in 5 sec.

11–1. Expressed in cgs units, the air-gap pull for an electromagnet is

$$F = \frac{\beta^2 A}{8\pi}$$

where F is in dynes, β in maxwells per square centimeter ($\beta = \phi/A$), and A is in square centimeters Change the expression to pounds kilomaxwells, and square inches

12-1. For Prob. 11-1 show that

$$F = \frac{\beta^2 A}{72{,}130{,}000}$$

where F is in pounds, β in maxwells per square inch, and area (A) in square inches

13-1. The air-gap ampere turns (NI) are determined from

$$NI = \frac{\beta l}{0.4\pi\mu}$$

where β is in maxwells per square centimeter, and l is in centimeters Express β in maxwells per square inch and the length (l) in inches

14-1. The expression for the capacity of a plate condenser in microfarads is,

$$C = \frac{kA}{4\pi d \times 9 \times 10^5}$$

where C is in microfarads, A is in square centimeters, d is in centimeters Write the expression with the plate spacing (d) in mils and the area in square inches

15-1. The resistance of a copper wire is expressed by

$$R = \frac{10.37 \times l}{A}$$

where R is in ohms, l is in feet, and A is in circular mils If one circular mil is $(\pi/4)\,(0.001)^2$ sq in , determine the expression for the resistance in absolute ohms, meters, and square centimeters

CHAPTER 2

ELECTROMOTIVE FORCE: CAUSE

1. Source. Electricity is a part of the realm of matter not created by man but transformed by man for practical use. If electrical potential can be so arranged that there is a gradient between two points, a *cause* (difference of potential) has been established in the natural sequence of phenomena; and, if the *opposition* can be overcome, there will be an *effect.*

Electromotive force, the difference of electrical potential, may be obtained in several ways—by:

1. Photoelectric action.
2. Thermal action.
3. Chemical action.
4. Electromagnetic induction.

The first two methods have only secondary application in the power field. Their generated electromotive force is so small that a large number of units would be required to produce an appreciable amount of energy, making the cost prohibitive for commercial use. However, both are important in electrical measurements and control, for their low potentials may be amplified and, when amplified, may be used to control large power circuits.

The third and oldest method of obtaining electromotive force has for many years played a very important part in the use of electricity. The primary battery is used in the field of energy engineering and in electrical measurements. The particular ability of the secondary battery to store electrical energy by chemical reaction makes this form of accumulator important, because energy from wind, water, or other intermittent sources can be stored until needed. At present, the chemical source finds use in ignition, automobile lighting, measurements, emergency lighting, control and stand-by service, communication, and signal service. The use of the chemical accumulator for power purposes is practically obsolete except in special cases.

Potential for the important field of power electrical engineering is obtained by the fourth method, electromagnetic induction. This type of generation feeds the power lines, which, like a giant spider web, cover

the whole United States and other countries. The greater portion of this book will deal with the transmission and control of this type of generated potential.

PRIMARY BATTERIES

2. Definition. A *primary battery* consists of *primary cells* designed to produce electric potential by means of an electrochemical reaction which is not sufficiently reversible to permit the cell, when discharged, to be efficiently recharged by an electric current. A *dry cell* is a cell in which the electrolyte is in the form of a jelly, which is absorbed in a porous medium, or is otherwise restrained from flowing from its intended position, so that the cell is completely portable and the electrolyte is non-spillable.

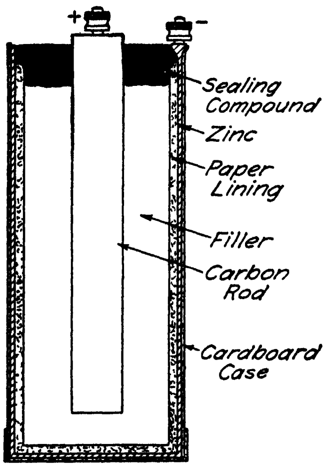


FIG. 1–2. Cutaway view of a dry cell showing the details of construction.

3. Dry Cell. The first type of primary cell was a wet cell. There are many combinations of metals and solutions that will give a difference of potential between the electrodes. This potential, when applied to a resistance, will cause a flow of current for a period of time, the length of this time period depending upon polarization. Polarization is an apparent increase in the internal resistance of the cell caused by increased contact resistance between the electrolyte and the electrode. The wet cell was the first source of voltage for the experimenter and, since a variety of possibilities existed, there appeared a great number of combinations, most of which are now obsolete. Modified forms of the wet cell are still used in experimental work, the Le Clanche cell being used most frequently.

This cell, in a modified form, appears on the market as the dry cell, shown in Fig. 1–2. An outer cardboard container serves a dual purpose as protection and as a medium for printed matter. The cell proper consists of a zinc cup lined with porous paper, and filled with a moist paste consisting of sal ammoniac, manganese dioxide (the depolarizing agent), and some form of powdered carbon, which increases the conductivity. Inserted in the center of the paste is the cathode, a carbon rod, the shape of which depends upon the manufacturer's design.

The zinc degenerates as the battery is used and, if the battery is permitted to lie idle, deterioration of the cell occurs through evaporation

of the moisture from the paste. The life of the cell is 12 to 18 months, with the voltage of the cell ranging from 1.5 volts to 1.4 volts and with an internal resistance of approximately 0.1 ohms. Heavy current may be drawn intermittently from the cell, but it will soon polarize if much current is drawn continuously. The cells, when connected positive to negative (in series), may be arranged to supply any desired voltage, where the total voltage is the summation of the individual voltages. The cells may be arranged with all positives and all negatives connected together to common leads (in parallel). The voltage will be equivalent to the voltage of one cell, but the total current supplied will be divided among the cells, thereby reducing the requirements from each cell.

The cost per unit of energy supplied by a dry cell is very high and, therefore, they are useful only for small amounts of power. Ignition systems, flashlights, radios, signal circuits, clocks, telephones, and measuring instruments are the logical applications. Though dry cells may be somewhat rejuvenated by perforating the outer case and setting them in water, it is more economical to discard them when they show a marked depreciation. An ammeter used in testing this type of cell should have a resistance, including the connecting wires, of 0.01 ohm or less.

SECONDARY BATTERIES

4. Definitions. A *storage battery* is a group of connected *electrolytic storage cells* which store electrical energy and which, when discharged, may be restored to a charged condition by causing an electric current to flow through them in a direction opposite to the flow of current when the cell is being discharged. There are two types of storage cells of interest to the engineer, the *lead-acid* cell and the *nickel-iron-alkaline* cell. The first is commonly spoken of as the lead cell; the latter as the Edison cell.

LEAD-ACID BATTERY

5. Construction. Figure 2–2 shows the construction of the lead-acid cell. There is an outer jar which is of glass in the stationary battery and of rubber in the portable battery. This jar holds a solution of sulphuric acid and distilled water in which two groups of lead plates, separated from each other by either perforated rubber, glass fiber, or grooved wood separators, are placed. In the portable battery, each cell has a hard rubber cover sealed to the cell proper with asphalt compound, and the adjacent cell terminals are burned together; in the stationary battery, the glass jars may be open and the connections are bolted together and

covered with petroleum grease to prevent corrosion. In the portable type, the cells are assembled in either a wood or, more frequently, a rubber case; in the stationary type, jars are placed in trays of clean sand to reduce conduction between cells.

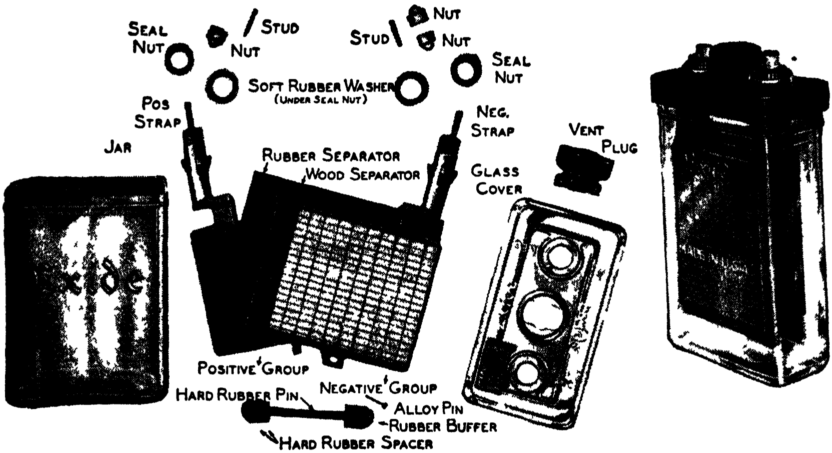


FIG. 2–2. Parts and an assembled lead-acid cell. (Courtesy of Electric Storage Battery Co.)

6. Plates. The plates are constructed with a large surface of active material which has low resistance, yet is strong enough to withstand the service to which the battery is subjected. There are two methods of forming the plates. One is the Planté method, in which the sheet of lead is grooved and the plate is formed by electrochemically converting the lead. This type of plate is large and strong and is the most satisfactory for stationary batteries. The second method for platė construction is the Fauré, in which a grid structure is filled with a prepared paste. In order that the grid may have structural strength the lead is alloyed with antimony. When this plate is formed, the active material and the grid together make a solid structure particularly suitable for the portable battery because it requires little space and withstands vibrations.

Like plates are grouped together and burned into a battery post, forming the positive and the negative groups. When the battery is fully charged, the positive plates are chocolate colored and the negative plates are gray.

7. Operation. When the battery is to be charged, it is connected as shown in Fig. 3–2, with the positive and negative terminals of the battery connected to the positive and negative lines, respectively. If the battery is to be charged from a voltage source much greater than the

battery terminal voltage, lamps or a resistor should be used for current control. To test the battery for charge, the specific gravity of the electrolyte is measured. For the stationary battery this varies from 1.18 to 1.22 (Baumé) and, for the portable battery, from 1.2 to 1.3 from

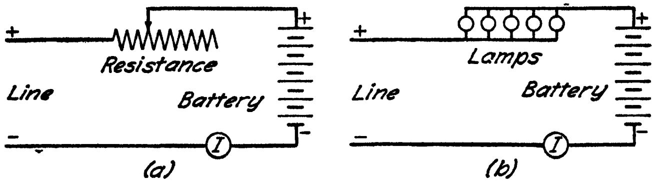


FIG. 3–2. Methods of charging storage batteries from a d-c line: (*a*) by using a rheostat; (*b*) by using a lamp bank.

discharge to charge. Frequently, other factors besides the condition of charge influence the specific gravity. Temperature is one such factor. Figure 4–2 shows the lowest specific gravity permitted for various degrees of temperature.

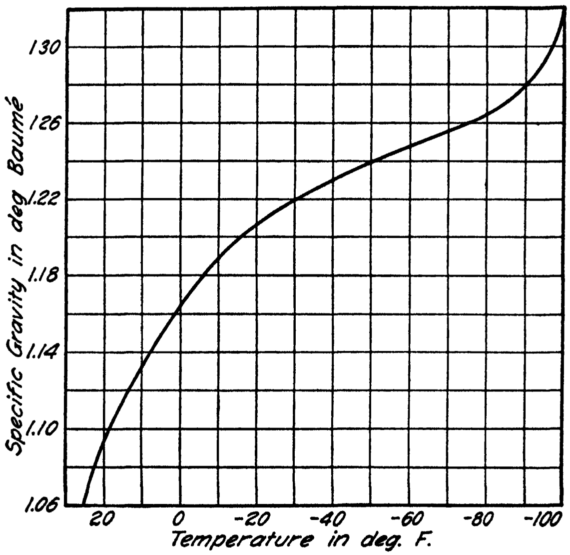


FIG. 4–2. Freezing curve for sulphuric acid electrolyte.

The rating of a battery is on an ampere-hour unit, but, unless there is a definite understanding regarding the time required for the discharge, the rating by this quantity has no meaning. Figure 5–2 shows charge and discharge characteristics of the lead-acid cell.

Most batteries are based on an 8-hour discharge rate with voltages dropping to 1.7333 as the end voltage (Navy Standard) for automobile batteries. However, a value of 1.75 volts is commonly accepted as the end voltage in making tests on portable storage batteries.

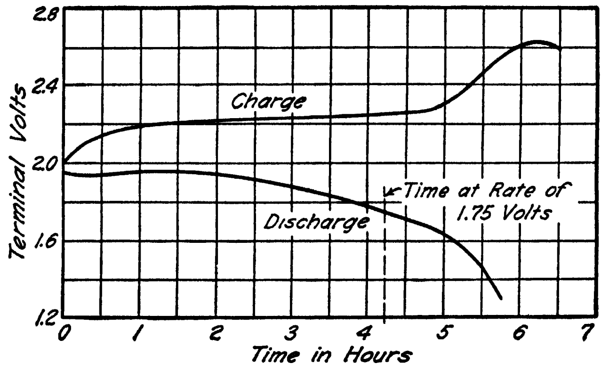


FIG. 5-2. Charge and discharge curves for a lead-acid cell, usually an 8-hr rate.

NICKEL-IRON-ALKALINE BATTERIES

8. Construction. The construction of the Edison cell is quite different from that of the lead-acid cell. The plates are not formed but are built up. The nickel or positive plate is a nickeled-steel grid containing spirally wound steel tubes filled with alternate layers of nickel hydroxide and flaked metallic nickel. The metallic nickel acts as a conductor to offset the increased resistance of the battery caused by the hydroxide changing to an oxide. These tubes are made so that they can expand to a limited degree. The iron (or negative) plate is also composed of a nickeled-steel grid with rectangular pockets instead of tubes. These pockets hold rectangular cells filled with powdered iron oxide.

The plates of each type are assembled into groups and fastened together with nickeled-steel rods. These groups are nested together, and separated from each other by hard rubber pins, and from the case by either spacers or hard rubber sheets. The groups are assembled with the negative plate on the outside; therefore, one more negative than positive plate appears in the assemblage. This assemblage is placed in a cold-rolled steel case. There is not a large sediment space left at the bottom of the cell, as there is in the lead-acid cell.

After the assembled plates are placed in the case, the cover is welded on and the terminals are sealed so that the electrolyte cannot spill out. As in the portable lead-acid cells, there is provision for the gas to escape

without loss of liquid. When the assemblage is complete, the cell is filled with potassium hydrate (caustic potash) of a 21 per cent strength in distilled water with a small amount of lithia added to lengthen the life of the cell and to increase its capacity. Figure 6–2 shows the plates and groups of the Edison cell in a cutaway view.

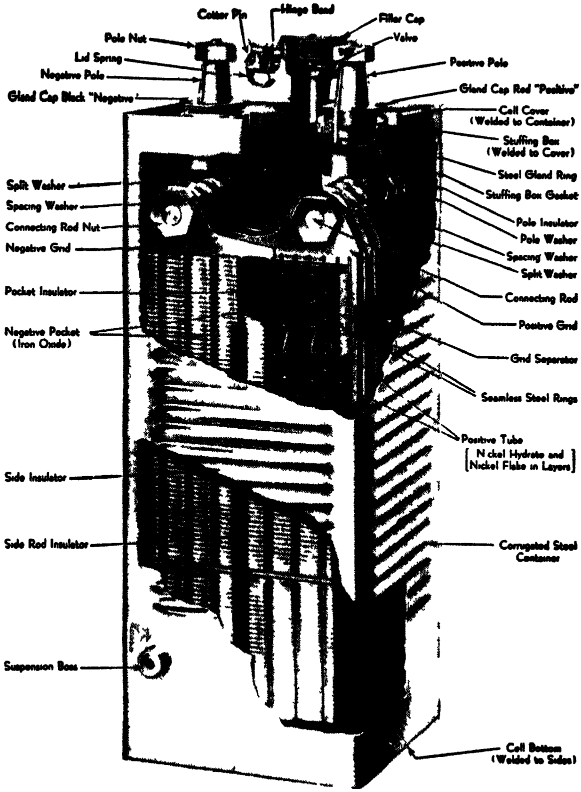


FIG. 6–2. Cutaway view of an Edison cell showing the details of construction (Courtesy of Edison Storage Battery Co.)

9. Operation. The operating voltage of the Edison cell ranges from 1.1 volts at discharge to 1.45 volts at charge. The characteristic curves are shown in Fig. 7–2. The ampere-hour rating is based on the discharge possible before an end voltage of 1.0 volt is reached and is normally tested at a 5-hour rate. When the battery is first used, it will increase in capacity until it reaches approximately 15 per cent in excess of its normal capacity, after which time it decreases until it is worn out, at about 80 per cent of its normal capacity.

This battery is very sturdy and will withstand severe use and neglect, partly because in the Edison cell the electrolyte acts merely as a conductor and is not an active agent as it is in the lead-acid cell.

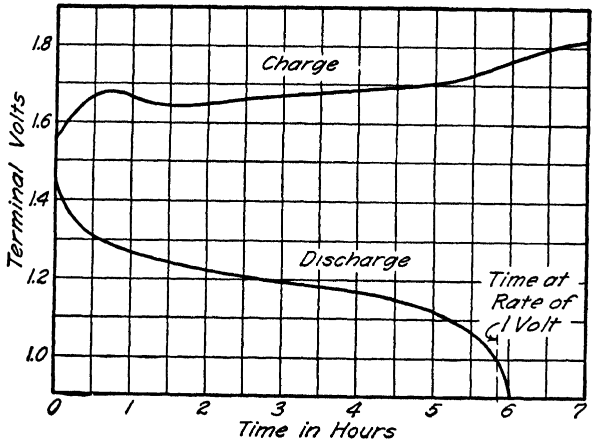


FIG. 7–2. Charge and discharge curves for an Edison cell, usually a 5-hr rate.

10. Comparison. A comparison of the advantages of the two types of batteries will, to some extent, determine the use to which they can be successfully adapted.

LEAD-ACID CELL	NICKEL-IRON-ALKALINE
Low cost	Long life
Low internal resistance	No harm in overcharging
Maintains voltage	No harm in reverse charging
High efficiency	Can be idle for long periods
Small temperature rise	Short circuit will not harm
Large capacity per unit volume	Vibration does not injure

11. Application of Storage Batteries. The storage battery is used where a constant voltage supply is needed (as in the calibration of instruments and precision measurements) for stand-by service in emergency or peak loads, for very large current demands at low voltage, and wherever portable power is necessary. Automobiles, industrial trucks, mine locomotives, and signal equipment need batteries to operate. In the communication field the battery is considered the most satisfactory source of voltage, though replaced by rectifiers and filters because of convenience.

MECHANICAL GENERATION OF ELECTROMOTIVE FORCE

12. Law of Electromagnetic Induction. The electromotive force induced in a circuit is proportional to the time rate of change of the magnetic flux linked with the circuit. This law is often associated with the name of Faraday, but was not formulated by him. However, in 1830 he discovered that an electromotive force is generated in a conductor when either the conductor cuts lines of force or lines of force cut the conductor.

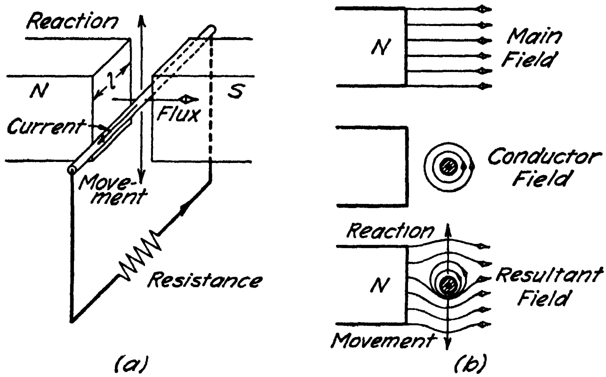


FIG. 8–2. Electromotive force generated by the movement of a conductor in the presence of a magnetic field.

When an induced electromotive force is obtained from a machine which transforms mechanical energy into electrical energy, the machine is called a generator regardless of whether the potential generated is alternating or direct. A-c machines produce periodic voltages having alternate positive and negative values, whereas the d-c machine produces a unidirectional voltage having approximately constant values, either negative or positive, depending upon the reference chosen.

The simplest form of generator is one in which a conductor is passed through a magnetic field, as shown in Fig. 8–2*a*. The reaction in such a machine may be easily determined by the right-hand rule. *When a current-carrying conductor is grasped in the right hand so that the thumb is in the direction of current flow, the fingers will indicate the direction of the lines of magnetic flux surrounding the conductor.* To determine the direction of current flow, the right-hand rule must be used in conjunction with Lenz's Law, which is: *When a closed circuit is moved in a magnetic field in such a way that a change takes place in the lines of magnetic induction passing through the circuit, a voltage is induced in the circuit causing*

a current to flow in such a direction that the mechanical force set up between it and the magnetic field opposes the motion. This, stated in a simpler form, means that induced currents flow in such a direction that their reaction (magnetic field created) tends to oppose the action which produces them, namely: the law of action and reaction as applied to the general physical laws. Figure 8–2*b* shows the action, reaction, flux, and current in proper relationship.

If the conductor shown in Fig. 8–2*a* is closed through a resistance and the conductor is moved downward, as shown, a current will flow in a direction which opposes the motion and establishes a reaction in the direction indicated. As a result, the field produced by the current will build up the magnetic field below the conductor. If lines of flux are

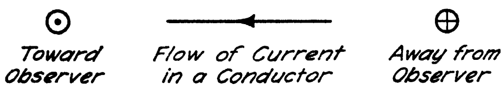


FIG. 9–2. Conventional representation of current flow in the electrical system.

added to the conductor to increase the number of lines below it, the current, according to the right-hand rule, must flow out of the figure. The symbols for the indication of current flow in a conductor are shown in Fig. 9–2. This method of approach will enable one to determine the reaction caused by current flow in the motor or in the generator, but the determination will depend more upon analysis than upon memory.

The voltage induced in a conductor moved through a magnetic field as shown in Fig. 8–2*a* may be expressed by

$$e = \beta l v \times 10^{-8}$$

for the induced electromotive force (e) is directly proportional to flux density (β lines per square centimeter), the length of the conductor cut by the flux (l in centimeters), and the velocity (v in centimeters per second), the electromotive force will be in abvolts, which when multiplied by 10^{-8} changes the quantity to the practical unit, the volt. The product of l and v is the area swept through per second and this area multiplied by the flux density is the total flux cut per second, or the induced electromotive force is proportional to the flux cut per second, the form of Faraday's Law used in Art. 14. It is assumed in the foregoing discussion that the movement of the wire and the flux with respect to the wire are at 90 degrees as shown in Fig. 8–2, the usual condition in the electromagnetic machine.

A generator of the simple type shown in Fig. 8–2 would not be practical because it would be necessary to have not only magnetic fields of

great length but also complicated forms of driving devices. The acyclic machine approaches this condition, although, in place of the field being elongated, the elements of a disk are brought into position in successive order. This form of machine would have a voltage characteristic similar to that of the battery.

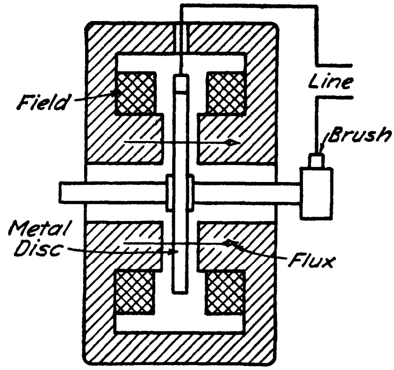


FIG. 10–2. Schematic diagram of an acyclic generator. A metallic disk rotated in a magnetic field.

13. Sources of Ripple-Free D-C Voltage. The battery and the acyclic generator are the only true sources of direct current. The acyclic machine shown in Fig. 10–2 is merely an application of the Faraday disk (1831) and generates a unidirectional current which, like that of the battery, is free from any ripple. This machine, again like the battery, is limited to low potential differences at reasonable costs and has found application in the electroplating industry and resistance welding where high currents at low voltage are demanded.

Figure 11–2 shows, in a diagram, the steady-state characteristics of the generated voltage of the battery for a short period of time and of the acyclic machine. The instantaneous voltage, after steady-state operation has been reached, is equal to the maximum voltage and is constant (assuming a constant load to be applied and the cell to be operated at voltages within its charged range). Another qualification of this type of electromotive force is that it is free from ripple, and this is essential in the operation of communication equipment.

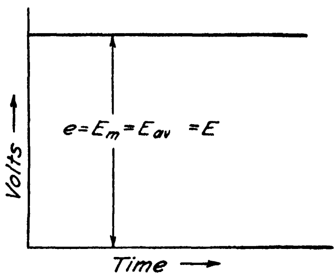


FIG. 11–2. Characteristic voltage curve for the battery for a short period, and the acyclic generator.

14. A-C Generator. Since the circumference of a circle is a path which, for a constant angular velocity, gives a constant linear velocity and is endless, it is logical that in the practical machine the conductor should travel such a path. Figure 12–2 shows a simple generator utilizing this principle, also adding the south pole of a magnet in close proximity to the north pole to improve the magnetic density. When the conductor is moved in front of the north pole, as in Fig. 12–2, the other

side of the loop is being moved in front of the south pole, and voltage is induced in it in the opposite direction. The voltages in the two sides of the loop are additive. The direction of current flow in the two sides of the loop may be determined by the right-hand rule. As soon as the two sides of the loop change places, there is again an induced voltage, but an investigation of current flow through the resistance shows that the current has reversed. This is a simple form of an a-c generator (or alternator). If certain basic assumptions are made, the character of this periodic wave may be accurately determined.

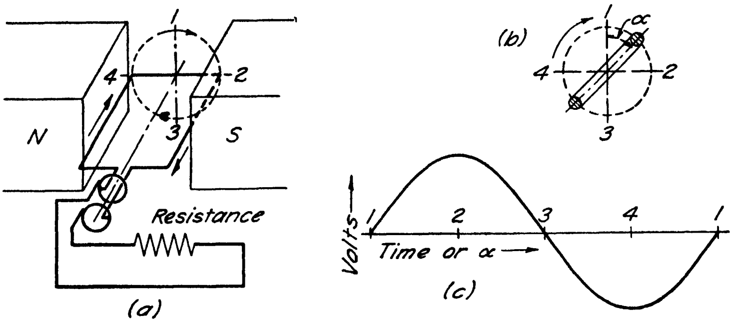


FIG. 12–2. Generation of alternating electromotive force (sine wave) when a loop of wire is rotated in a magnetic field.

The absolute volt is the electromotive force induced in a coil of one turn when the flux threading the coil is changing at the rate of 1 line per second. The volt (the practical unit) is the electromotive force induced in a coil of one turn when the flux threading the coil is changing at a rate of 10^8 lines per second. The negative sign is used when the electromotive force is increasing in a positive direction and the flux is decreasing $(-d\phi/dt)$. The induced voltage causes a current flow which tends to prevent this decrease of flux.

A mathematical statement of the Law of Electromagnetic Induction is

$$e = -N\frac{d\phi}{dt} \times 10^{-8} \quad \text{or} \quad -\frac{d(N\phi)}{dt} \times 10^{-8} = -\frac{d\lambda}{dt} \times 10^{-8} \qquad (a\text{–}2)$$

where e is the induced electromotive force in volts, N is the number of turns in the coil, and $d\phi/dt$ is the rate of change of flux with respect to time t in seconds. The value $N\phi$ is known as the flux linkage (λ) and the usual procedure is to express the induced electromotive force by the rate of change of linkages. Since the flux threads the coil and the coil surrounds the flux, the two are linked together in a manner similar to the

links of a chain. The product of the units of flux and the turns is the numerical value for flux linkage (λ).

If the coil in Fig. 12–2*b* is assumed to be in a magnetic field of constant magnitude, the lines of flux linking the coil at any instant will be a cosine function of the maximum flux:

$$\phi_i = \phi_m \cos \alpha$$

This substituted in the original form gives

$$e = -N \frac{d}{dt} (\phi_m \cos \alpha) \times 10^{-8} = -N \frac{d}{dt} (\phi_m \cos \omega t) \times 10^{-8}$$

If the coil is revolving at a uniform angular velocity of ω, the angle $\alpha = \omega t$ and the value 10^{-8} occurs when the units are changed from absolute to practical units. Evaluation of the expression gives

$$e = N\phi_m\omega \sin \omega t \times 10^{-8} \qquad \omega t, \text{ radians}$$

or

$$e = N\phi_m\omega \sin 360ft \times 10^{-8} \quad 360ft, \text{ electrical degrees}$$

where f is the frequency and in which e is a maximum value when $\sin \omega t$ is unity. Then, E_m equals ($N\phi_m\omega\ 10^{-8}$) and by substitution the instantaneous voltage will be

$$e = E_m \sin \omega t = E_m \sin \alpha = E_m \sin 360ft$$

This will be recognized as a form of harmonic motion, a periodic quantity. The *frequency* of the periodic quantity, in which time is the independent variable, is the number of periods occurring in unit time, where the *period* is the smallest value of the increment of the independent variable which separates recurring values of the quantity.

In electrical engineering the term *degree* is not applied to the angular turning as in mathematics; an *electrical degree* is the 360th part of the angle subtended, at the axis of a machine, by two consecutive field poles of like polarity. One *mechanical degree* is thus equal to as many electrical degrees as there are pairs of poles on the machine. Reference to Fig. 12–2 shows that a conductor passing from the position 1 through 2, 3, 4 and back to 1 (two consecutive field poles) generates one sine wave. The abscissa can be scaled in either time (t) or degrees, the angle being 360 electrical degrees. For the two-pole machine the mechanical and electrical degrees will be the same; for the four-pole machine there will be 360 mechanical degrees but 720 electrical degrees since two sine waves will be generated in one revolution.

In the expression (a–2) the instantaneous voltage is considered. Frequently it is the maximum or the average voltage that is desired. This may be obtained by using the following expressions:

$$E_m = N\phi_m\omega \times 10^{-8} \tag{b-2}$$

$$E_{av.} = \frac{N\phi}{t} \times 10^{-8} = \frac{N(\phi_2 - \phi_1)}{t_2 - t_1} \times 10^{-8} \tag{c-2}$$

where ϕ/t is the average rate of change of flux, N is the number of turns in the coil, ϕ_m is the maximum flux threading the coil, ϕ is the total change of flux, ω is the angular velocity in *electrical radians* per second ($2\pi f$) (expressed as $360f$ electrical degrees), t is the time necessary in seconds to produce the change in flux, which in a machine is the time necessary to pass through 90 electrical degrees. For a machine, $180p$ will be the number of electrical degrees in a machine having p poles.

$$s = \frac{\text{rpm}}{60} \text{ revolutions per second}$$

$$t = \frac{1}{2ps}$$

where T, the time in seconds, is

$$T = 4t$$

or

$$T = \frac{2}{ps}$$

Example a. A coil consisting of 100 turns of wire is revolving at the rate of 1800 rpm in a constant magnetic field of a two-pole machine. If the maximum flux passing through the coil is 200,000 lines, what will be the instantaneous voltage generated in the coil when the rate of change is a maximum? What is the average voltage?

$$e = -N\frac{d\phi}{dt} \times 10^{-8}$$

$$E_m = N\phi_m\omega \times 10^{-8}$$

$$\omega = \frac{P}{2} \times \frac{\text{rpm}}{60} \times 2\pi$$

$$\omega = \frac{1800 \times 2\pi}{60} = 188.496 \text{ radians per second for a two-pole machine}$$

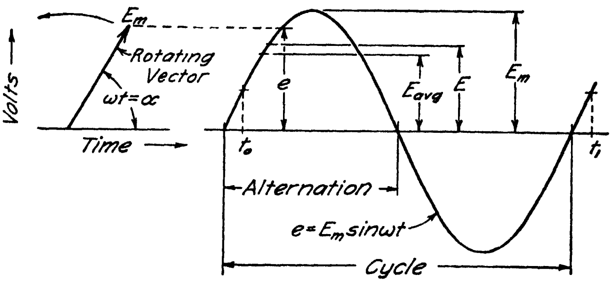
$$E_m = \frac{100 \times 200{,}000 \times 188.496}{10^8} = 37.8 \text{ volts}$$

$$E_{av.} = \frac{N\phi}{t} \times 10^{-8}$$

$$t = \frac{1}{2ps} = \frac{1}{2 \times 2 \times 30} = \frac{1}{120}$$

$$E_{av.} = \frac{100 \times 200{,}000 \times 120}{10^8} = 24 \text{ volts}$$

15. Sine Wave Definitions. (Refer to Fig. 13–2.) There have been two basic assumptions in the previous discussion: first, the angular velocity is uniform; second, the magnetic field is uniform. Though the first is easily obtained in practice, the latter is only approximated and



f = frequency
E_m = maximum emf
e = instantaneous emf
E = effective emf
E_{av} = average emf

$\frac{1}{T} = \frac{1}{T_1 - T_0}$ = frequency
ωt = angular displacement
$\omega = 2\pi f$ = angular velocity
$\frac{1}{f} = T = T_1 - T_0$ = period

FIG. 13–2. The sine wave and its characteristic definitional quantities.

seldom will the actual machine give a smooth sine wave. Since in all elementary work the analysis of circuits and machines assumes the sine wave or sinusoidal conditions, a thorough acquaintance with a number of definitions based on the sine wave is necessary.

Period. The period of a periodic quantity is the smallest increment of the independent variable which separates recurring values of the quantity.

Frequency. The frequency of a periodic quantity, in which time is the independent variable, is the number of periods occurring in unit time.

Angular Velocity. The angular velocity of a periodic quantity is the frequency multiplied by 2π. If the periodic quantity can be considered as resulting from the uniform rotation of a vector (Fig. 13–2), the angular velocity is the number of radians per second passed over by the

rotating vector. A simple sinusoidol quantity can be represented by a vector (see p. 122).

Cycle. The cycle is the complete series of values of a periodic quantity which occur during a period.

Alternation. An alternation is half of a cycle.

16. Frequency. Figure 12–2 shows that, for the two-pole machine, each revolution generates a single cycle or that there is one cycle per pair of poles; therefore,

$$f = \frac{p}{2} \times \frac{\text{rpm}}{60}$$

where p is the number of poles, rpm is the revolutions per minute, and f is the frequency. The standard frequencies in the United States are 60 and 25 cycles, with some 40- and 50-cycle systems in the West. Since it is impossible to have single or part poles, there is a definite speed for any combination of frequency and poles.

Example b. What will be the frequency of the alternating voltage generated by an 8-pole machine with a speed of 900 rpm? At what speed must the machine be driven to generate voltage at 25 cycles per second?

$$f = \frac{p}{2} \times \frac{\text{rpm}}{60}$$

(*a*)
$$f = \tfrac{8}{2} \times \tfrac{900}{60} = 60 \text{ cycles}$$

(*b*)
$$\text{rpm} = \frac{120 \times 25}{8} = 375 \text{ rpm}$$

17. Average Value of a Sine Wave. The average value of a periodic quantity is the algebraic average of the values of the quantity taken through one period and is expressed by

$$Y_{\text{av.}} = \frac{1}{T}\int_0^T y\,dt$$

if y is the periodic function of t and T is the period. Application of this definition to the sine wave gives zero as a result for the complete cycle. This definition for the sine wave has been modified to include only a half-period average, and this is expressed by

$$Y_{ha} = \frac{2}{T}\int_0^{T/2} y\,dt$$

the half-period value of a symmetrical alternating quantity.

Figure 14–2 shows the sine wave and the graphical representation of the average value with the ordinates used in the determination of the average value.

Using equation

$$Y_{ha} = \frac{2}{T}\int_0^{T/2} y\,dt$$

and applying it to the sine waves,

$$E_{\text{av.}} = \frac{2}{2\pi}\int_0^{2\pi/2} e\,d\alpha$$

$$e = E_m \sin \alpha$$

$$E_{\text{av.}} = \frac{E_m}{\pi}\int_0^{\pi} \sin \alpha\,d\alpha$$

$$= \frac{E_m}{\pi} - \left(\cos \alpha\right)_0^{\pi} = \frac{E_m}{\pi} - [(-1) - (1)] = \frac{2E_m}{\pi}$$

$$= 0.636E_m$$

The half-period average value of a sine wave is 0.636 times the maximum value.

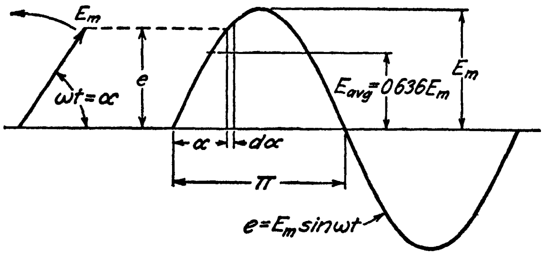


FIG. 14–2. Determination of the average value for the sine wave electromotive force.

Example c. What would be the average value of electromotive force ~~if the~~ generated voltage wave is $e = 100 \sin \omega t$?

$$e = E_m \sin \omega t$$

$$E_m = 100$$

$$E_{\text{av.}} = 0.636E_m$$

$$= 0.636 \times 100 = 63.6 \text{ volts}$$

18. Effective Value of a Sine Wave. The effective value of a periodic quantity is the square root of the average of the squares of the quantity taken throughout one period. If y is a periodic function of t,

$$Y_{\text{eff.}} = \left(\frac{1}{T}\int_0^T y^2\,dt\right)^{1/2}$$

where $Y_{\text{eff.}}$ is the effective value of y, and T is the period.

The effective value has been defined so that both direct and alternating currents will have the same magnitude, and so that the conversion of electrical to heat energy will be the same in a pure resistance, regardless of whether an ampere of direct or alternating current is passing through the resistance.

$$I_{\text{d-c}}{}^2R = (\text{average of } i_{\text{a-c}}{}^2)R$$

$$I_{\text{d-c}} = \sqrt{\text{average of } i_{\text{a-c}}{}^2} = I_{\text{eff.}}$$

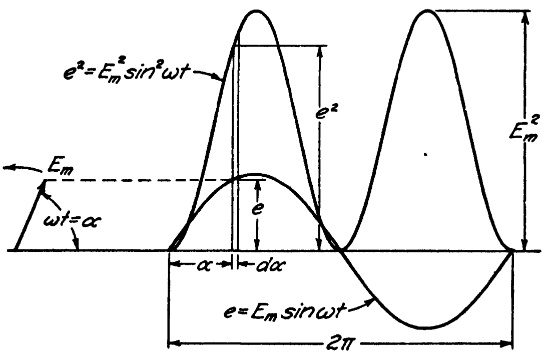


FIG. 15–2. Determination of effective value for the sine wave electromotive force.

If the general expression is applied to the sine wave (Fig. 15–2) the effective value may be determined as follows:

$$E_{\text{eff.}} = \left(\frac{1}{2\pi}\int_0^{2\pi} e^2\, d\alpha\right)^{1/2}$$

$$e = E_m \sin \alpha$$

$$e^2 = E_m{}^2 \sin^2 \alpha$$

$$E_{\text{eff.}} = \left(\frac{E_m{}^2}{2\pi}\int_0^{2\pi} \sin^2 \alpha\, d\alpha\right)^{1/2}$$

$$= \left[\frac{E_m{}^2}{2\pi}\left(\frac{\alpha}{2} - \frac{\sin 2\alpha}{4}\right)_0^{2\pi}\right]^{1/2}$$

$$= \left[\frac{E_m{}^2}{2\pi}\left(\frac{2\pi}{2} - \frac{\sin 4\pi}{4} + \frac{\sin 0°}{4}\right)\right]^{1/2}$$

$$= \left(\frac{E_m{}^2}{2} - 0\right)^{1/2} = \frac{E_m}{\sqrt{2}} = 0.707E_m$$

The effective or root-mean-square (r.m.s.) value for a sine wave is 0.707 times the maximum value of e. This is the value read by the voltmeter. The unit has been established in this manner, making it the same in both a-c and d-c systems. The effective and maximum values in the d-c system are the same. The effective value is the one used in all discussion dealing with problems concerning a-c circuits and machines.

Example d. What will be the effective value of the alternating voltage which has a wave form given by the equation $e = 50 \sin 10t$?

$$E_m = 50$$

$$E = 0.707E_m$$

$$E = 35.4 \text{ volts}$$

Example e. What would be the effective value of a periodic wave form which is composed of semicircles having radii of 50 units? $e = \sqrt{100\alpha - \alpha^2}$, $e = 0$ when $\alpha = 0$ and 100.

$$E = \left(\tfrac{1}{100}\int_0^{100} e^2\,d\alpha\right)^{1/2}$$

$$= \left(\tfrac{1}{100}\int_0^{100} (100\alpha - \alpha^2)\,d\alpha\right)^{1/2}$$

$$= \left(\tfrac{1}{100}\int_0^{100} 100\alpha\,d\alpha - \tfrac{1}{100}\int_0^{100} \alpha^2\,d\alpha\right)^{1/2}$$

$$= \left[\frac{100}{100}\left(\frac{\alpha^2}{2}\right)_0^{100} - \frac{1}{100}\left(\frac{\alpha^3}{3}\right)_0^{100}\right]^{1/2}$$

$$= \left[\frac{100}{100}\left(\frac{100^2}{2} - \frac{0}{2}\right) - \frac{1}{100}\left(\frac{100^3}{3} - \frac{0}{3}\right)\right]^{1/2}$$

$$= \left(\frac{100^2}{2} - \frac{100^2}{3}\right)^{1/2}$$

$$E = \sqrt{1667} = 40.8 \text{ units}$$

19. Other Values of Wave Shapes. The foregoing definitions may be applied to any other type of wave form encountered in the electrical engineering field, but in elementary courses only the sine wave is to be considered. The average value is used in the determination of the form factor of a wave, and this form factor is defined as the ratio of the effective value of the quantity to its half-period average value. For the sine wave, it follows that

$$\text{Form factor} = \frac{0.707E_m}{0.636E_m} = 1.11$$

For the sine wave, this has little meaning but, for a wave which is not sinusoidal, it is a descriptive factor which gives an index coefficient for wave shape and is never less than unity.

20. D-C Generator. The d-c generator operates in the same manner as the a-c generator described in Art. 14, generating an alternating wave electromotive force in the winding. The distinct difference in the two machines is in the delivery of the voltage to the outside circuit. In the a-c machine, the wave form appearing in the outer circuit is the same as that in the winding, having been delivered through slip rings but, in the d-c machine, one-half the wave is reversed by means of a device known as a commutator.

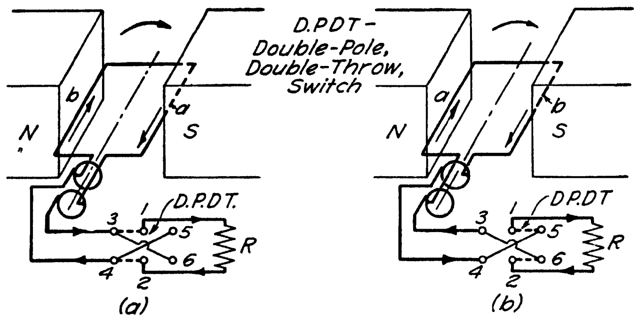


FIG. 16–2. The rectification of the generated electromotive force (sine wave) to a unidirectional (d-c) wave form by means of a commutating switch.

The type of electromotive force available from the battery as a source is very desirable in many applications, but the high cost of this energy prohibits extensive installations for power purposes. Though the d-c generator does not have the ripple-free characteristic of the battery or acyclic generator, it approaches such a condition and for practical purposes can be used in most installations to replace the expensive source represented by a battery.

For a simple demonstration, a double-pole, double-throw switch is connected, as shown in Fig. 16–2, to the alternating supply furnished by the generator winding. With the coil side *a* under the south pole, the current flows outward and, with the commutating switch in the position shown (Fig. 16–2*a*), the current flows into the resistance *R* as indicated by the arrow. When the coil side *a* moves to a position under the north pole, the current flow is reversed and would be reversed in the resistance *R* if the commutating switch had not been reversed but, with the reversal of this switch (Fig. 16–2*b*), the current continues to flow in the same direction. A switch properly connected to the revolving

armature would do this automatically. This is accomplished by substituting a commutator for the double-throw switch.

21. Wave Form of the D-C Machine. If a sectional ring (which is a commutator) instead of slip rings is placed on the armature, the effect is the same as when the commutating switch is used. The lower loop

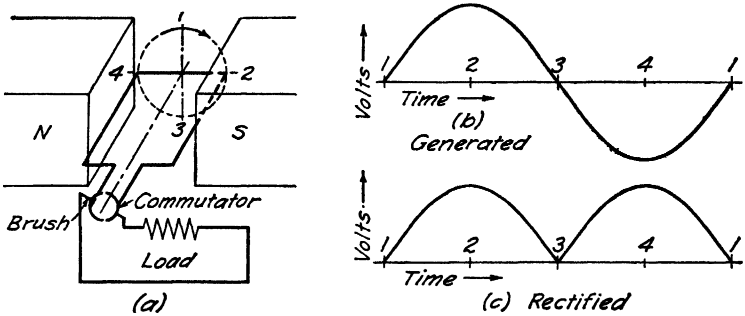


FIG. 17–2. (*a*) The generated sine wave rectified by means of a commutator. (*b*) Generated sine wave. (*c*) Rectified unidirectional wave.

of the sine wave will be reversed and the whole wave form will appear above the base line as shown in Fig. 17–2*c*. This wave form is far from the steady supply associated with direct current, and any study of this type of wave would be very involved. Waves of this form are found in ignition systems and full-wave rectifiers. If, however, several windings are placed at different angular positions on the armature and connected

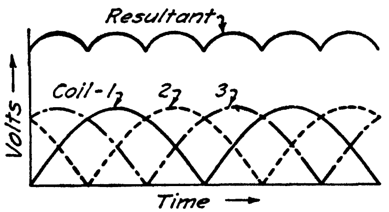


FIG. 18–2. Resultant rectified wave when three coils at 60° on the armature are connected in series.

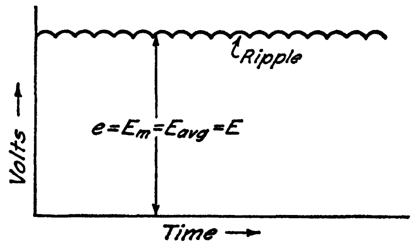


FIG. 19–2. Resultant wave form for a d-c generator showing a ripple which is so small a portion of the whole that for most purposes the wave factors are equal.

in series, the result will be an addition of instantaneous voltages and the curve will then appear with a ripple as shown in Figs. 18–2 and 19–2. It is impossible to remove the ripple completely without an infinite number of coils, but it may be reduced to an insignificant quantity in the practical machine by the addition of enough coils spaced around the

armature. The d-c generator produces a form of direct current different from that obtained from a battery or acyclic machine, in that there is a ripple of high frequency superimposed upon the average steady voltage. This ripple in the commercial machine is within audio frequencies and, for that reason, the d-c generator cannot be used in communication application without a filter for removing this disturbance.

22. D-C Wave Form Constants. If the d-c machine is compared with the a-c machine in wave form, it is seen that the various quantities are all constant and equal (Fig. 19–2).

$$e = E_m$$

$$E_{av.} = E = E_m$$

23. Summary. Since the sine wave of the a-c machine and the constant potential of the d-c machine are basic for all discussions of the distribution of electrical power and operation of the machines, their individual characteristics are tabulated in Table I–2. The potential from batteries would be treated the same as the d-c machine.

TABLE I–2

COMPARISON OF THE PRINCIPAL COEFFICIENTS IN THE A-C AND D-C SYSTEMS [1]

	(Sinusoidal) Alternating Current	Direct [2] Current	Battery
Instantaneous value	$e = E_m \sin \alpha$	$e = E_m$	$e = E_m$
Average value	$E_{av} = 0.636E_m$	$E_{av.} = E_m$	$E_{av} = E_m$
Effective value	$E = 0.707E_m$	$E = E_m$	$E = E_m$
Form factor	1.11	1	1

[1] This comparison is not rigorous but simplifies the analysis of electricity and will be satisfactory for engineering purposes. It is usual to base a-c concepts upon sine waves and d-c concepts upon a constant generated electromotive force. To compare vectors for direct and alternating current is not possible, but it is satisfactory to express both the alternating and direct current in symbolic forms (complex quantities) and compare them upon this base. This considers direct current as a special case of alternating current.

[2] Ripple neglected.

PROBLEMS

1–2. A 2-pole generator, having 50 turns on the armature, produces a maximum of 18.9 volts when the pole strength is 10^5 lines. What is the speed of the generator?

2–2. A sine wave is generated by a 4-pole, 1000 turn, generator having a pole strength of 10^6 lines. The coil moves from a position of zero flux to 50 per cent flux in 0.05 sec. Determine the average voltage for this interval.

3–2. A 2-pole, 500-turn generator produces a sine-wave voltage expressed by $e = 15.72 \sin 377t$. (*a*) What is the maximum pole strength? (*b*) What is the frequency of the voltage generated?

4–2. A sine wave voltage with a maximum of 37.7 volts is generated at a frequency of 60 cycles when 50 turns are placed on the armature. Determine (*a*) the maximum pole strength and (*b*) the instantaneous voltage when the coil is 45° from the maximum voltage position.

5–2. Time is taken at zero value, for a sine wave, at the 30° point of the wave with the voltage increasing in a positive direction. What will be the instantaneous voltage for the 50-cycle wave having a maximum value of 100 volts at the time 0.01 sec?

6–2. Given an equilateral triangular wave having a maximum ordinate of 200 volts, determine (*a*) maximum value, (*b*) average value, and (*c*) effective voltage.

7–2. List in tabular form the average and effective voltage, in terms of the maximum voltage, for these waves: (*a*) triangular, (*b*) isosceles trapezoidal (base 3 times parallel side), (*c*) the wave form for a half-wave rectified sine wave, (*d*) the wave form for a full-wave rectified sine wave.

8–2. When a 2-pole sine-wave generator, having 1000 turns and revolving at 2000 rpm, causes a voltmeter to deflect 100 volts, what is the pole strength of the machine?

9–2. The field of a 2-pole sine wave generator is adjusted to a value of 2×10^5 lines and driven at 1200 rpm. If the armature has 1000 turns, what will be (*a*) the maximum value of the instantaneous voltage, (*b*) the effective value, and (*c*) the average value?

10–2. Two generators are constructed to be assembled on a common shaft, each having the same number of poles and turns on its armature. Generator 1 produces an isosceles-trapezoidal wave form (base 3 times parallel side) with a maximum voltage of 10 volts when the pole strength is 5×10^6 lines. If at the same time generator 2 produces a sine-wave voltage with a maximum of 10 volts what is the field strength of generator 2?

11–2. A circuit generates the wave form of an isosceles trapezoid at half-wave rectification. The parallel edges of the trapezoid have a ratio of 3 to 1. When the voltage maximum is $E_m = 141.4$ volts, what will be the average and effective values?

12–2. (*a*) Determine the revolutions per minute necessary to generate a sine-wave average voltage of 240 volts with a 2-pole machine having a flux of 2×10^6 maxwells and 500 turns on the armature. (*b*) What is the maximum voltage?

13–2. A 6-pole sine-wave generator is driven by a 1200-rpm motor, has 150 turns on the armature, and operates with a flux of 2×10^5 maxwells. Determine (*a*) the maximum, (*b*) average, and (*c*) effective voltage.

14–2. A wave form consists of an isosceles triangle with $E_m = 100$ volts from 0 to π and a rectangular shape from π to 2π with $E_m = -100$ volts. Determine the average and effective voltage for the wave form.

15-2. Two generators connected in series having the following voltages phased with a common reference are

$$E_1 = 39 \text{ volts, at an angle } \tan\theta_1 = 1.2$$

$$E_2 = 62 \text{ volts, at an angle } \tan\theta_2 = 2.26$$

Both machines are on a common shaft operating at a frequency of 50 cycles. How many turns must be placed on the armature of a 4-pole machine to give an equivalent voltage if the field has a flux of 2×10^5 maxwells?

CHAPTER 3

RESISTANCE: ENERGY-CONSUMING COUNTERACTION

1. Definitions and Analogies. There are three physical parameters, namely, resistance, inductance, and capacitance, which impede the flow of current in an electric circuit. Each parameter has specific characteristics and each impedes the flow of electric current differently. This chapter will discuss the circuit parameter, resistance, and its characteristics.

"Resistance is the (scalar) property of an electric circuit or of any body that may be used as part of an electric circuit which determines for a given current the rate at which electric energy is converted into heat or radiant energy and which has a value such that the product of the resistance and the square of the current gives the rate of conversion of energy.

"In the general case, resistance is a function of the current, but the term is most commonly used in connection with circuits where the resistance is independent of the current." *

Resistance may be compared, in its counteraction to the flow of current, to friction in the fields of mechanics and hydraulics. For example, when a wooden block is being moved uniformly on a smooth horizontal plane, the energy necessary to keep the block moving is used to overcome the effects of friction. When an automobile moves along the highway, the engine supplies energy through the drive shaft to the rear wheels and this energy is used in overcoming the opposition of windage and mechanical friction. Another example occurs in the flow of water from the central pumping station to the consumer. The difference in head or pressure at the station and the pressure at the outlet on the consumer's premises is caused by the friction of the pipes.

All energy-consuming opposition to motion of water, mechanical motion, or to the flow of current is influenced by two things: distance of travel and cross-sectional area. The opposition is directly proportional to distance and inversely proportional to area.

2. Units of Resistance. Resistance is measured in several systems of units, each stating, by definition, a basis for determining the value of these units.

* From American Standard Definitions of Electrical Terms—1941.

a. Abohm. The CGS Electromagnetic Unit of Resistance. "The c.g.s. unit of resistance is the resistance of a conductor when, with an unvarying current of one abampere flowing through it, the potential difference between the ends of the conductor is one abvolt." One abvolt is 10^{-8} volt.

b. The International Ohm. "The international ohm is defined as the resistance at zero degrees centigrade of a column of mercury of uniform cross-section, having a length of 106.300 centimeters and a mass of 14.4521 grams." Experimental results show that 1 international ohm equals 1.00048 absolute ohms.

c. Ohm. The practical unit of resistance is the ohm [one billion (10^9) abohms] and is the resistance of a circuit or of any body permitting a uniform current of one ampere to flow when a uniform voltage of one volt is impressed across the circuit.

Two other units of resistance often used are the megohm and the microhm. The megohm (10^6 ohms) is used when referring to high resistances and, because of the developments in the radio field, it is widely used. The megohm is used also in giving the resistances of insulation materials. The microhm (10^{-6} ohm) is used in expressing low resistances such as the resistances of bus bars, heavy current feeders, cables, and large conducting bodies.

3. Specific Resistance or Resistivity. From experimental observations, by comparing the effects of varying the length and cross-sectional area of a conductor, the relationship for resistance is

$$R = \rho \frac{l}{A}$$

where R = resistance in ohms
l = length of the conductor
A = cross-sectional area
ρ = resistivity or specific resistance per unit volume of material.

The physical dimensions of a conductor will affect its opposition to the flow of current. The calculation of the opposition becomes rather involved if the conductor is not uniform in cross-section, unless a replacement (or substitution) of an equivalent uniform conductor is made. In the field of electrical engineering, the conductors are uniform and the analysis of the factors controlling resistance is based upon this condition.

The specific resistance or resistivity coefficient, which is dependent upon the kind of material used, is an important factor influencing the resistance of a conductor. If the cgs system is used, the specific resistance factor (ρ) is the resistance offered between parallel faces by a specimen 1 square centimeter in cross-section and 1 centimeter long and

it is expressed in ohms per centimeter, cubed. If the practical system is used, the value of ρ may be expressed in ohms per square inch-foot; per circular mil-foot; or per inch, cubed. Conductors having the same cross-section and length but made from iron, steel, copper, silver, or aluminum offer different oppositions to the flow of current.

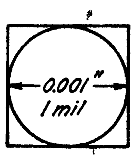


FIG. 1-3. Comparison of the circular mil area to the square mil area.

The unit most commonly used in expressing the volume of a conductor is the circular mil-foot, meaning a cross-section of one circular mil and a length of one foot.

A circular mil is the area of a circle, the diameter of which is 1 mil (one thousandth of an inch). Reference to Fig. 1-3 shows that the unit circular mil area is smaller than the square mil area. This can be shown also by mathematical consideration since the area of the circle in square mils is

$$\frac{\pi D^2}{4}(1_{\square})$$

$(1_{\square})$ is the definitional unit area and in this particular case it is the square mil.

The area in circular mils $= D^2(1_{\bigcirc})$.

$(1_{\bigcirc})$ is the definitional unit area and in this particular case it is the circular mil. D in each case is the diameter expressed in mils.

If the area is the same for both considerations,

$$\frac{\pi D^2}{4}(1_{\square}) = D^2(1_{\bigcirc})$$

and

$$(1_{\bigcirc}) = \frac{\pi}{4}(1_{\square}) \quad \text{or} \quad (1_{\bigcirc}) = 0.7854(1_{\square})$$

The unit circular mil = 0.7854 square mil.

A change in physical dimensions will cause a change in the resistance of a conductor. For example, consider two conductors of the same material, shown in Fig. 2-3. The resistance along the length l for conductor b is one-half the resistance of conductor a since b is two times the area of a and thus the current will encounter one-half the opposition. Consider the case of the two conductors of the same material, as shown in Fig. 3-3. In this case the resistance of b is two times that of a as b is two times as long as a.

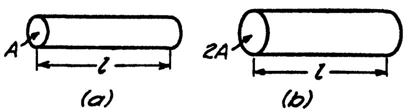


FIG. 2-3. The resistance of a conductor as affected by area.

If Figs 2–3*b* and 3–3*b* are referred to, it is observed that the volume of the material in the two examples is the same. However, the resist-

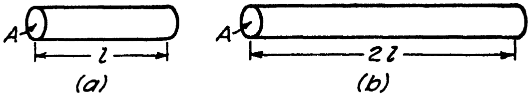


FIG. 3–3. The resistance of a conductor as affected by length.

ance in the latter is four times that in the former. This can be developed in the following way:

Since $V = lA$, $A = V/l$, and $l = V/A$, substituting in $R = \rho(l/A)$ gives the final form

$$R = \rho \frac{l}{\frac{V}{l}} = \rho \frac{l^2}{V} = \rho \frac{V}{A^2}$$

The resistance of a conductor having a given specific resistance and constant volume varies inversely as the square of the cross-section and directly as the square of the length. If the resistance of a circular mil-foot of a wire is known, it is possible to obtain its entire resistance by direct proportional relationship.

$$R = \rho \times \frac{\text{feet}}{\text{circular mils}}$$

where ρ is the specific resistance in ohms per circular mil-foot.

From Table I–3 the value of ρ for copper is 10.37 ohms. Therefore the previous expression, when copper wire is being considered, is

$$R = \frac{10.37 \times \text{feet}}{\text{circular mils}} \text{ ohms} \qquad \text{—copper}$$

The value of ρ for iron is 60.09, and iron wire would have

$$R = \frac{60.09 \times \text{feet}}{\text{circular mils}} \text{ ohms} \qquad \text{—iron}$$

Example a. What is the value of resistance between opposite sides of a 1-in. cube of copper? Use Table I–3 for the value of ρ.

From Table I–3, the resistance per cubic centimeter for copper is 1.724 microhms. Since the resistance for a given material varies as $\frac{l}{A}$, the resistance of the cube is $R = 1.724 \times \frac{2.54}{2.54 \times 2.54}$ or $R = \frac{1.724}{2.54} = 0.679$ microhm.

Example b. If the cube of copper in the previous example is rolled until its cross-section is reduced to 0.2 sq in., what is its resistance between the surfaces having the small areas?

TABLE I-3

SPECIFIC RESISTANCE—RESISTIVITY

Material	Resistivity at 20° C	
	Resistance per Cubic Centimeter in Microhms (10^{-6} ohm)	Resistance per Circular Mil-Foot in Ohms
Metals		
Aluminum	2.828	17.01
Copper	1.724	10.37
Gold	2.44	14.7
Iron	10.00	60.09
Lead	22.03	132.3
Mercury	95.75	576.0
Silver	1.61	9.65
Tungsten	5.6	33.5
Alloys		
Brass	7.1	42.2
Manganin (Cu 84%, Mn 12%, Ni 4%)	44	266
Nichrome	100	612

The volume in the two instances, before and after rolling, is the same.

$$R = \rho \frac{V}{A^2}$$

Therefore

$$\frac{R_1}{R_2} = \frac{\rho \dfrac{V_1}{A_1^2}}{\rho \dfrac{V_2}{A_2^2}} \quad \text{or} \quad \frac{R_1}{R_2} = \frac{A_2^2}{A_1^2}$$

Substituting

$$\frac{0.679}{R_2} = \frac{(0.2)^2}{(1.0)^2}$$

$$R_2 = \frac{0.679}{0.04}$$

$$R_2 = 16.97 \text{ microhms}$$

Example c. What is the resistance of 1000 ft of copper wire with a cross-section of 106,000 (A.W.G.-No. 0) cir mils?

$$R = \frac{10.37 \times 1000}{106,000} = 0.0978 \text{ ohm}$$

Example d. How many feet of iron wire, with a cross-section of 10,000 cir mils, would be required to have a resistance of 5 ohms?

$$5 = \frac{60.09 \times \text{feet}}{10,000}$$

$$\text{Length in feet} = \frac{50,000}{60.09} = 833 \text{ ft}$$

4. Conductivity. The reciprocal of the specific resistance or resistivity is called the conductivity. The values of resistivity are a measure of the opposition offered by various conductors, whereas the values of conductivity are a measure of the abilities of the conductors to carry current. Since copper is generally used as a conductor, the conductivity of annealed pure copper is chosen as a standard for comparison and its conductivity is taken as 100 per cent. This comparison of conductivity can be made on a basis of equal volumes or equal masses. The international Annealed Copper Standard is given for the standard temperature of 20 degrees centigrade. The standard in terms of volume resistivity is 10.371 ohms per mil-foot and in terms of mass resistivity is 0.15328 ohm per meter-gram. Table I–3 shows the value of specific resistance for copper to be 10.37 ohms per circular mil-foot. Therefore, the volume conductivity of silver, using copper as 100 per cent, is found to be $(10.37/9.65) \times 100$, or 107.3 per cent. Thus, considering volume conductivity, silver is a better conductor of current than copper.

Another comparison of the conductivities can be made, with the weight or mass as a basis. This second consideration is of importance in the design of transmission lines and in vertical distribution systems for the tall modern buildings. The mechanical problem involved in designing a transmission line requires careful attention to the weight of the conductor. Also, the weight of the conductor is affected by the current it must carry and its permissible resistance. If the specific resistance of a conducting material of unit length and unit weight has four times the specific resistance of another material of similar length and weight, the mass conductivity of the first is 25 per cent of the second. As in volume conductivity, the value of annealed pure copper is used for comparison. The resistance at 20° C of a uniform conductor 1 meter long and weighing 1 gram is 0.15328 ohm.

The values of volume conductivity and mass conductivity of a naterial can be expressed in percentage of annealed pure copper as

(a) $$\text{Volume conductivity} = \frac{10.37}{R_v} \times 100 \text{ per cent}$$

(b) $$\text{Mass conductivity} = \frac{0.15328}{R_m} \times 100 \text{ per cent}$$

where R_v is resistance per circular mil-foot and R_m is resistance per meter-gram, all values at 20° C.

5. Variation of Resistance with Temperature. In nearly every instance, the ohmic resistance of materials changes with a change in temperature, this change varying in both magnitude and sign for different materials. Most of the metallic materials show an increase in resistance with an increase in temperature, whereas most of the nonmetallic materials show a decrease in resistance for an increase in temperature. As an example of the latter class, glass, considered a good insulator under normal temperatures, is a very poor insulator when heated to a red heat.

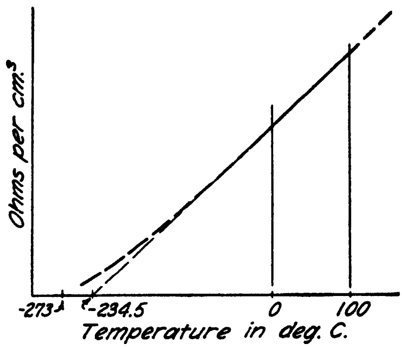


FIG. 4-3. Effect of temperature on resistance.

It has been observed experimentally that the ohmic resistance variation is practically a straight line for a reasonable change in temperature (−10° to 100° C). Referring to the curve of resistance as a function of temperature (Fig. 4-3), the extension of the curve to give zero resistance would be some temperature $-T°$ centigrade, provided the relationship between resistance and temperature remained constant. Assuming that the variation in most instances will be between 0° and 100° C, an equation of the form

$$y = ax + c$$

will express this relationship. If the resistance R_{T_1} is known, the resistance R_{T_2} can be calculated from the expression

$$R_{T_2} = R_{T_1}[1 + \alpha_1(T_2 - T_1)]$$

in which α_1 † is the temperature coefficient of resistance expressed at the

† The temperature coefficient of resistance is the percentage change in resistance per degree change in temperature.

temperature T_1. The temperature coefficient at another temperature can be expressed as

$$\alpha_2 = \frac{\alpha_1}{1 + \alpha_1(T_2 - T_1)}$$

The value of the coefficient α is usually given for the temperatures 0° and 20° C. If the coefficient α for 0° C is used, the above expression becomes

$$R_{T_2} = R_0(1 + \alpha_0 T_2)$$

TABLE II-3

TEMPERATURE COEFFICIENT OF RESISTANCE

Material	Temperature Coefficient	
	0° C	20° C
Aluminum	0.00423	0.0039
Copper	0.00427	0.00393
Iron	0.00624	0.0055
Platinum	0.0039	0.00367
Tungsten	0.0049	0.0048
Manganin	......	0.000006
Nichrome	. .	0.00039
Carbon (lamp filament)		−0.0009

Example e. The resistance of a coil of copper wire is 60 ohms at 0° C. What is its resistance when the temperature is increased to 47° C?

From Table II-3 the value of α at 0° C is 0.00427. Thus

$$R_{47°} = 60[1 + (47 \times 0.00427)]$$

$$R_{47°} = 72.04 \text{ ohms}$$

Example f. The resistance of a carbon pile resistor at 20° C is 1.2 ohms. What is its resistance after the current flowing through the resistor has increased its temperature to 65° C?

From Table II-3 the value of $\alpha_{20°}$, for carbon, is −0.0009.

The value of $R_{65°} = 1.2[1 - 45(0.0009)]$.

$$R_{65°} = 1.15 \text{ ohms}$$

Changes in resistance which are caused by temperature variations are apparent in the performance of generators and motors. The change in the field resistance as the machines continue to operate causes the field current to change and affect the performance. After a few hours of continuous operation, the machine reaches a constant temperature and no further change in resistance is observed.

6. Wire Tables. Since most electrical conductors are circular, a designation for indicating by gage number the resistance, cross-section, weight, and current capacity has been developed for use in England and the United States. Although several gage systems have been developed, the one most widely used is the American Wire Table (Brown and Sharpe). This table is based on the circular mil as the unit of cross-sectional area because of the ease with which areas and diameters may be expressed. The development of the wire table is based on a constant ratio between diameters of wires having successive gage numbers. Taken in order of gage number, the diameters are in a geometrical progression. Likewise, the areas of successive gage numbers have a definite ratio, the value being the square of the ratio of diameters. The larger the gage number, the smaller the diameter and area. Originally the gage number represented the number of times the wire was drawn through the dies to obtain that size. Under the present methods of manufacture, however, this is no longer true; but the same gage numbers, diameters, and areas have been retained.

The ratio of diameters can be found by making the following comparison. The diameter of gage No. 0 is 0.3249 inch and gage No. 40 is 0.003145 inch. Between No. 0 and No. 40 are 40 gage numbers. Therefore the ratio of any one diameter to the next diameter (large gage-number—smaller diameter and area) can be indicated as

$$\sqrt[40]{\frac{0.3249}{0.003145}} = \sqrt[40]{103.3068} = 1.123$$

The ratio of areas becomes $(1.123)^2 = 1.261$. The value of $(1.261)^3$ is 2.0049, and it follows, therefore, that the area is approximately either doubled or halved for every 3 gage numbers. Also, for every 10 gage numbers, the ratio of areas is $(1.261)^{10}$ or 10.1 approximately. Thus, in round numbers, the relationship for areas is 1.26 for successive gage numbers, 2 for every 3 gage numbers, and 10 for every 10 gage numbers.

It is comparatively simple to approximate closely the values in the wire table by using the values of resistance per 1000 feet, diameter, and area for No. 10 wire as a basis. For example, if No. 10 wire has 10,000 circular mils area, 1 ohm per 1000 feet resistance, and diameter of 100 mils, determine the corresponding values for No. 4. No. 4 wire is 6 sizes larger than No. 10; therefore the area is $(1.26)^6$ or $(2)^2$, or 4 times as large as No. 10, which is 40,000 circular mils. The resistance per 1000 feet for No. 4 is $\frac{1}{4}$ that of No. 10 or 0.25 ohm, and the diameter is $(40{,}000)^{\frac{1}{2}}$ or 200 mils.

Example g. What is the resistance, area, and diameter of No. 23 wire? Use values for No. 10 as a basis.

TABLE III-3. WIRE TABLE

TABLE OF STANDARD ANNEALED COPPER WIRE
AMERICAN WIRE GAGE (B & S)

Complete Wire Table—Theoretical Values

AWG	Diameter, Inches			Area, Circular Mils	Weight, lb per 1000 ft	Length, ft per lb	Resistance at 68° F		
	Min.	Nom.	Max.				Ohms per 1000 ft	Ft per ohm	Ohms per lb
0000	0.4554	0.4600	0.4646	211,600.	640.5	1.561	0.04901	20,400.	0.00007652
000	.4055	.4096	.4137	167,800.	507.9	1.968	.06180	16,180.	.0001217
00	.3612	.3648	.3684	133,100.	402.8	2.482	.07793	12,830.	.0001935
0	.3217	.3249	.3281	105,500.	319.5	3.130	.09827	10,180.	.0003076
1	.2864	.2893	.2922	83,690.	253.3	3.947	.1239	8,070.	.0004891
2	.2550	.2576	.2602	66,370.	200.9	4.977	.1563	6,400.	.0007778
3	.2271	.2294	.2317	52,640.	159.3	6.276	.1970	5,075.	.001237
4	.2023	.2043	.2063	41,740.	126.4	7.914	.2485	4,025.	.001966
5	.1801	.1819	.1837	33,100.	100.2	9.980	.3133	3,192.	.003127
6	.1604	.1620	.1636	26,250.	79.46	12.58	.3951	2,531.	.004972
7	.1429	.1443	.1457	20,820.	63.02	15.87	.4982	2,007.	.007905
8	.1272	.1285	.1298	16,510.	49.98	20.01	.6282	1,592.	.01257
9	.1133	.1144	.1155	13,090.	39.63	25.23	.7921	1,262.	.01999
10	.1009	.1019	.1029	10,380.	31.43	31.82	.9989	1,001.	.03178
11	.08983	.09074	.09165	8,234.	24.92	40.12	1.260	794.	.05053
12	.08000	.08081	.08162	6,530.	19.77	50.59	1.588	629.6	.08035
13	.07124	.07196	.07268	5,178.	15.68	63.80	2.003	499.3	.1278
14	.06344	.06408	.06472	4,107.	12.43	80.44	2.525	396.0	.2032
15	.05650	.05707	.05764	3,257.	9.858	101.4	3.184	314.0	.3230
16	.05031	.05082	.05133	2,583.	7.818	127.9	4.016	249.0	.5136

17	0.04481	0.04526	0.04571	2,048.	6.200	161.3	5.064	197.5	0.8167
18	.03990	.04030	.04070	1,624.	4.917	203.4	6.385	156.5	1.299
19	.03553	.03589	.03625	1,288.	3.899	256.5	8.051	124.2	2.065
20	.03164	.03196	.03228	1,022.	3.092	323.4	10.15	98.5	3.283
21	.02818	.02846	.02874	810.1	2.452	407.8	12.80	78.11	5.221
22	.02510	.02535	.02560	642.4	1.945	514.2	16.14	61.95	8.301
23	.02234	.02257	.02280	509.5	1.542	648.4	20.36	49.13	13.20
24	.01990	.02010	.02030	404.0	1.223	817.7	25.67	38.96	20.99
25	.01770	.01790	.01810	320.4	.9699	1,031.	32.37	30.90	33.37
26	.01578	.01594	.01610	254.1	.7692	1,300.	40.81	24.50	53.06
27	.01406	.01420	.01434	201.5	.6100	1,639.	51.47	19.43	84.37
28	.01251	.01264	.01277	159.8	.4837	2,067.	64.90	15.41	134.2
29	.01115	.01126	.01137	126.7	.3836	2,607.	81.83	12.22	213.3
30	.00993	.01003	.01013	100.5	.3042	3,287.	103.2	9.691	339.2
31	.008828	.008928	.009028	79.7	.2413	4,145.	130.1	7.685	539.3
32	.007850	.007950	.008050	63.21	.1913	5,227.	164.1	6.095	857.6
33	.006980	.007080	.007180	50.13	.1517	6,591.	206.9	4.833	1,364.
34	.006205	.006305	.006405	39.75	.1203	8,310.	260.9	3.833	2,168.
35	.005515	.005615	.005715	31.52	.09542	10,480.	329.0	3.040	3,448.
36	.004900	.005000	.005100	25.00	.07568	13,210.	414.8	2.411	5,482.
37	.004353	.004453	.004553	19.83	.06001	16,660.	523.1	1.912	8,717.
38	.003865	.003965	.004065	15.72	.04759	21,010.	659.6	1.516	13,860.
39	.003431	.003531	.003631	12.47	.03774	26,500.	831.8	1.202	22,040.
40	.003045	.003145	.003245	9.888	.02993	33,410.	1,049.	0.9534	35,040.
41	.00270	.00280	.00290	7.8400	.02373	42,140.	1,323.	.7559	55,750.
42	.00239	.00249	.00259	6.2001	.01877	53,270.	1,673.	.5977	89,120.
43	.00212	.00222	.00232	4.9284	.01492	67,020.	2,104.	.4753	141,000.
44	.00187	.00197	.00207	3.8809	.01175	85,100.	2,672.	.3743	227,380.
45	.00166	.00176	.00186	3.0976	.00938	106,600.	3,348.	.2987	356,890.
46	.00147	.00157	.00167	2.4649	.00862	116,000.	4,207.	.2377	488,010.

Courtesy of Belden Mfg. Co., Chicago.

No. 10 has resistance of 1 ohm per 1000 ft, area of 10,000 cir mils, diameter of 100 mils.

No. 20 is 10 sizes from No. 10, and it has a resistance of 10 ohms per 1000 ft; area of 1000 cir mils; diameter of $(1000)^{1/2}$ or 31.6 mils.

Then No. 23, being 3 gage numbers from No. 20, has a resistance of 20 ohms per 1000 ft; area of 500 cir mils; diameter of $(500)^{1/2}$ or 22.4 mils.

7. Resistors and Resistor Alloys. Resistors are placed in a circuit for the purpose of controlling the current or for producing heat. The design and construction must always allow for dissipating the heat generated. Field rheostats and controllers for motors are examples of resistors for current control, whereas electric furnaces, ranges, and toasters are examples of resistors for producing heat.

The field rheostats and controllers for motors and generators are heavily constructed and designed for years of service. This type of resistor, however, is seldom operated above 200° C and the disintegration and oxidation, usually experienced with higher temperatures, are reduced. The heat-producing resistors may be operated at high temperatures but, unless sufficient radiating surface is provided for those units which are operating continuously, the resistors are short-lived.

German silver was one of the first resistor alloys. This alloy, consisting of copper, nickel, and zinc in various percentages, is usually listed in accordance with its percentage of nickel and is purchased on this basis. The greater the nickel content in the alloy, the greater is the specific resistance. It also has a comparatively high temperature coefficient of resistance.

In present-day practice, the iron-nickel alloys are used for resistor materials whenever the operating temperatures are low. For general use, the copper-nickel alloys are used; and, when materials of high resistivity are desired, the chromium-nickel alloys are used. All these alloys are practically non-oxidizing at reasonable temperatures.

8. Insulating Materials. In direct contrast to the resistors previously mentioned, there exists a group of materials which have extremely high specific resistivity coefficients. These materials, because of their extremely low conductivity, permit very small or negligible currents to flow and, as a result, are used as insulators. These insulating materials may be used as mechanical supports for high conductivity materials in order to keep the current confined within the desired electric circuit.

Insulating materials may be solids, liquids, or gases. Examples of each class are (1) solids—glass, porcelain, rubber, mica, bakelite, (2) liquids—transformer oil, (3) gases—air. In the present distribution systems of electrical power, all three classes of insulating materials are used although the solids are most widely used. The voltage generated

in the generating station is produced in conductors insulated from each other with solids such as mica, paper, and cloth. The transformer, insulated with solids and liquids, converts the voltage to a suitable value for use in the home. The conductors, on insulators and separated by air (gas), carry the electrical power to the service entrance in the home. It must be remembered that no material has an infinite specific resistance and, for that reason, there is no perfect insulator available. However, many materials have high enough specific resistivity coefficients so that the leakage currents flowing may be neglected in engineering calculations.

9. Relationship Between Voltage, Current, and Opposition. When an electromotive force is applied across an opposition, a current which is directly proportional to the electromotive force and inversely proportional to the opposition will flow in the completed circuit. At the time Ohm's Law was first stated, the only known opposition to the current flow was resistance, but the use of alternating current presented two other kinds of opposition which must be considered in determining the counteraction to current flow. This relationship may be expressed as $E = IZ$ where Z is a factor which depends upon all three types of opposition. If, however, resistance is the only opposition present, Ohm's Law for this condition may be expressed as $E = IR$ since $Z = R$. These values are all expressed in the practical system of units.

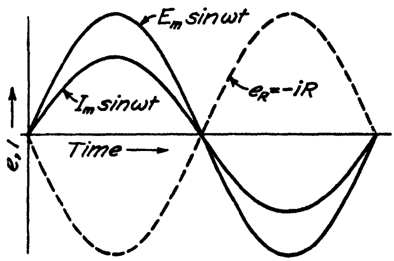


FIG. 5–3. Sine waves of current and voltage for a circuit containing pure resistance.

If the electromotive force is produced by an a-c generator, its expression will have the form $e = E_m \sin \omega t$. Ohm's Law can be written for this alternating voltage in a pure resistance as $e = iR$, where e and i are the instantaneous values of voltage and current at any time t. If the voltage

$$e = E_m \sin \omega t = E_m \sin 360ft$$

the current

$$i = \frac{E_m}{R} \sin \omega t = \frac{E_m}{R} \sin 360ft$$

or

$$i = I_m \sin \omega t = I_m \sin 360ft$$

the current $I_m \sin \omega t$ will flow in this circuit which contains only resistance and will always be in time phase with the voltage impressed on the circuit. This can be represented in graphical form (Fig. 5–3). The

current in the resistance R produces a voltage which is equal in magnitude and opposite in sign to the impressed voltage,

$$e = iR \qquad e_R = -iR$$

This voltage of opposition (e_R), if it were the only source available, would cause a current to flow in the direction opposite to the present flow of current. The impressed voltage and the voltage of opposition are indicated in Fig. 5-3.

10. Resistances in Series and in Parallel. When several resistances are connected in series, Fig. 6-3, the current is the same in each resistance. If the current is a

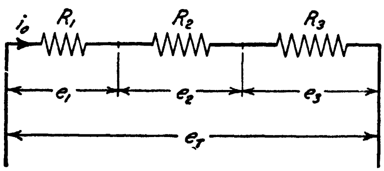


FIG. 6-3. A circuit consisting of resistances connected in series.

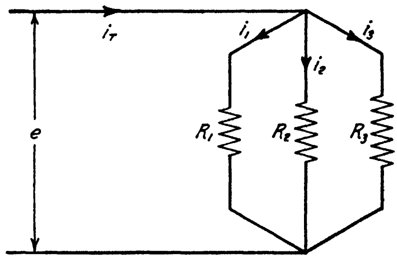


FIG. 7-3. A circuit consisting of resistances connected in parallel.

sinusoidal current ($i = I_m \sin \omega t$) the voltages for each resistance become

$$e_1 = R_1 I_m \sin \omega t$$

$$e_2 = R_2 I_m \sin \omega t$$

$$e_3 = R_3 I_m \sin \omega t$$

The total voltage

$$e_T = e_1 + e_2 + e_3$$

By substitution

$$e_T = R_1 I_m \sin \omega t + R_2 I_m \sin \omega t + R_3 I_m \sin \omega t$$

$$e_T = (R_1 + R_2 + R_3) I_m \sin \omega t$$

$$e_T = R_0 I_m \sin \omega t$$

where

$$R_0 = R_1 + R_2 + R_3$$

Therefore in a series circuit containing only resistances, the total resistance is equal to the numerical sum of the individual resistances of the circuit.

When several resistances are connected in parallel, Fig. 7-3, the voltage across each resistance is the same voltage. If the voltage across

the resistances is sinusoidal ($e = E_m \sin \omega t$) the current in each resistance becomes

$$i_1 = \frac{E_m \sin \omega t}{R_1}$$

$$i_2 = \frac{E_m \sin \omega t}{R_2}$$

$$i_3 = \frac{E_m \sin \omega t}{R_3}$$

The total current

$$i_T = i_1 + i_2 + i_3$$

and

$$i_T = \frac{E_m \sin \omega t}{R_1} + \frac{E_m \sin \omega t}{R_2} + \frac{E_m \sin \omega t}{R_3}$$

or

$$i_T = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) E_m \sin \omega t$$

If the voltage (e) and the total current (i_T) are considered as acting on one resistance only (in this case the equivalent of three resistances in parallel) then

$$e = R_0 I_{Tm} \sin \omega t$$

or

$$i_T = \frac{E_m \sin \omega t}{R_0}$$

but

$$i_T = i_1 + i_2 + i_3$$

Therefore

$$\frac{E_m \sin \omega t}{R_0} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) E_m \sin \omega t$$

and

$$\frac{1}{R_0} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

In a parallel circuit containing only resistances the reciprocal of the total resistance is equal to the sum of the reciprocals of the individual resistances which constitute the parallel circuit.

11. Power. At the beginning of this chapter, resistance was defined as a scalar quantity and also the part of an electrical circuit which determines the rate of energy transfer into heat. Of the three parameters of

the electric circuit, resistance is the only one which consumes power from the system. This average rate of power consumption is expressed in watts and is equal to I^2R, where I is the effective value of current. This expression is for average power and, if the power at any time t is desired, the instantaneous power will be vi where v and i are the instantaneous values of voltage and current respectively at the time selected. Since the current and voltage are in time phase for resistance, the instantaneous power can be expressed also as i^2R. In the a-c system where a current $i = I_m \sin \omega t$ is flowing, the average power is I^2R.

A summary of the power consumed in an electrical circuit can be given as

(1) Instantaneous power i^2R watts

(2) Average power I^2R watts

In the d-c system, the instantaneous and effective values of current are the same and, for every condition, the power is always I^2R.

Example h. A d-c voltage of 110 volts is impressed across a resistance of 10 ohms. How much current flows? How much power is consumed?

$$I = \frac{110}{10} = 11 \text{ amp}$$

$$P = (11)^2 \times 10 = 1210 \text{ watts}$$

Example i. A 60-cycle a-c voltage $e = 100 \sin \omega t$ is impressed on a resistance of 10 ohms. What is the instantaneous power when $t = 0$; $t = 1/240$ sec? What is the average power?

When $t = 0$, $e = 0$, $i = 0/10 = 0$, and instantaneous power is zero.

When $t = 1/240$ sec, $e = E_m = 100$, $i = 100/10 = 10$ amp, and instantaneous power is $(10)^2 \times 10 = 1000$ watts.

Average power $= I^2R$.

$$I = \frac{\frac{E_m}{\sqrt{2}}}{R}$$

$$I = \frac{70.7}{10} = 7.07 \text{ amp}$$

Average power $= (7.07)^2 \times 10 = 500$ watts.

PROBLEMS

1-3. Determine the resistance of a copper bus bar which is 3/8 in. by 3 in. by 6 ft.

2-3. What is the resistance of a 2500-turn coil wound with No. 20 copper wire if the mean length of turn is 8 in.?

3-3. What is the ratio of the resistance of 1500 ft of No. 4 copper conductor to 1200 ft of No. 2 aluminum conductor?

4–3. A coil is wound with 1600 ft of No. 17 copper wire. The coil is connected to 20 volts d-c. How much current will flow?

5–3. A resistance of 20 ohms is connected to a voltage $v = 200 \sin (377t + 15°)$. Determine (*a*) the sinusoidal expression for the current, (*b*) the reading of the ammeter connected in the circuit, (*c*) the power consumed.

6–3. A resistance of 25 ohms is connected to a 125-volt, 50-cycle, a-c supply. If the voltage is $v = V_m \sin \omega t$, determine the instantaneous power when $t = 0.005$ sec.

7–3. If the operating temperature of a tungsten filament of a 100-watt, 120-volt lamp is 2000° C, at rated voltage, how much current flows when the lamp is first connected to a 120-volt source (cold temperature 20° C)? Assume the temperature coefficient constant over the range given.

8–3. A current $i = 25 \sin (377t - 30°)$ flows in a circuit containing a resistance of 10 ohms. What is the sinusoidal expression for the voltage impressed, average power consumed, and instantaneous power when $t = 0.1$ sec?

9–3. A sinusoidal voltage and a symmetrical triangular voltage having the same maximum voltage of 150 volts are each connected to 10 ohms resistance. Compare the average power for the two sources.

10–3. The usual laboratory leads used with a voltmeter are 5 ft long and are of No. 18 copper insulated wire. If the leads are connected to a 125-volt bus and accidentally shorted together at the other end, how much current flows in the leads? Assume an infinite bus of 125 volts.

11–3. A shunt field of a d-c machine requires a d-c voltage of 108 volts to cause a current of 1 amp to flow at 20° C. If the current is maintained constant for 2 hr, the voltage required must be increased to 125 volts. Compute the rise in temperature for the coil. Assume a temperature coefficient of resistance at 0° C to be 0.004.

12–3. A voltage $v = 50 \sin [100\pi t - (\pi/3)]$ is applied to a 5-ohm resistance. (*a*) What is the average power in the circuit? (*b*) If the circuit consists of No. 12 copper wire, what is the length of the conductor?

13–3. A d-c circuit consisting of No. 14 copper wire extends from a 100-volt supply to a load 200 ft distant. The circuit becomes short circuited at the load. (*a*) What current flows? (*b*) What power is converted into heat?

14–3. A voltage $v = 100 \sin [100\pi t + (\pi/3)]$ is applied to a resistance of 10 ohms. (*a*) What is the instantaneous power when $t = 0.01$ sec? (*b*) What is the average power?

15–3. A 60-cycle symmetrical triangular voltage having a maximum value of 100 volts is connected to a 10-ohm resistance. If $t = 0$ when the voltage wave is passing through zero and increasing in a positive direction, determine the instantaneous power required for this resistance when $t = 0.0125$ sec.

CHAPTER 4

INDUCTANCE: REVERSIBLE COUNTERACTION

Another opposition characteristic of the electrical system is the inductance which, though it has the ability to oppose the flow of current, is operative only when there is a change of current.

1. Nature of Inductance. Inductance is the (scalar) property of an electrical circuit (or of two or more neighboring circuits) which determines the electromotive force induced in one of the circuits by a change of current in either of them. This characteristic gives rise to one of the oppositions, or counteractions, of the electrical circuit which, though it does not consume energy, is like resistance in that it is a part of the system and is measured by the dimensions of that part of the system.

Inductance bears the same relationship to the electrical system as inertia bears to the mechanical system. Inductance comes into action only when there is a change of current; inertia operates only when there is a change of motion; and, in both, energy is stored or returned to the system during the change. The energy consumed by inductance in any complete cycle is zero.

Reference to the characteristics of the generated voltages from the battery, d-c generator, and a-c generator in Chapter 2 shows that the battery will cause a flow of current which does not change and is free from ripple. In such a system, inductance will act only when a circuit is closed or opened or when the current is increased or decreased because of load change. In the d-c generator there is a slight ripple but this is such a small percentage of the whole that the effect is negligible and the inductance acts in the same way as in a battery circuit. In an a-c system, the current is changing many times per second with the result that inductance is an active factor in determining the opposition to the current flow in a system.

Before making a detailed study of the property of inductance, it will be well to inspect the definitions for inductance, some of which are more descriptive than the one previously given. Inductance may be defined as a coefficient of proportionality between the rate of change of current and the counterelectromotive force, written as

$$e = -\mathcal{L}\frac{di}{dt}$$

The negative sign occurs because the electromotive force opposes the change of the current which produces it.

The coefficient of inductance depends upon the flux linkages per unit current. This coefficient of self-inductance is usually spoken of as inductance. In this definition, the inductance is expressed as

$$\mathfrak{L} = \frac{d(N\phi)}{di} 10^{-8}$$

where $N\phi$ or λ is the number of linkages. If the flux (ϕ) is considered proportional to the current, the coefficient of self-inductance remains constant.

Example a. Determine the inductance of a coil that has 1000 turns, an average flux of 10^6 lines, and a current of 10 amp.

$$\mathfrak{L} = \frac{1000 \times 10^6}{10 \times 10^8} = 1 \text{ henry}$$

It is necessary to review the physical properties of magnetic fields to understand the function of inductance in an electrical system, since the above definitions show clearly that there is a definite relationship between the current in an electrical circuit and the magnetic field. This relationship is measured in units of that property of the circuit which brings into the system an opposition to the change of the electric current.

MAGNETICS

2. Magnetism. This is a characteristic of certain materials and electrical circuits which lacks a definite description and is frequently described by a physical demonstration of the ability to attract iron filings. Ancient peoples recorded this phenomenon as a property of certain stones, later assigned to a group known as the natural magnets or lodestones. There is no practical use for these in the electrical field. Magnetism became of practical value when, as a result of the discovery that these lodestones were capable of indicating directions, the compass was invented.

The development of artificial magnets, of both a permanent and a temporary type, parallel the development of knowledge in electrical engineering. The electrical system, with all its electrical apparatus, is dependent upon magnetism and magnets, and a study of the underlying principles of magnetics, the branch of science dealing with the laws of magnetic phenomena, is essential to an understanding of the operation of electrical systems and apparatus.

3. The Magnetic Field. This, as generally used, indicates the region throughout which the magnetizing forces are of significant magnitude with respect to the conditions under consideration. This field is represented conventionally by lines of force surrounding the body or coil possessing the magnetic properties.

Figure 1-4 shows the magnetic field surrounding a bar magnet and a coil (solenoid) carrying current. These field plots may be readily taken with iron filings if a piece of cardboard is placed over the magnet or in the coil carrying current and iron filings are sifted upon the surface.

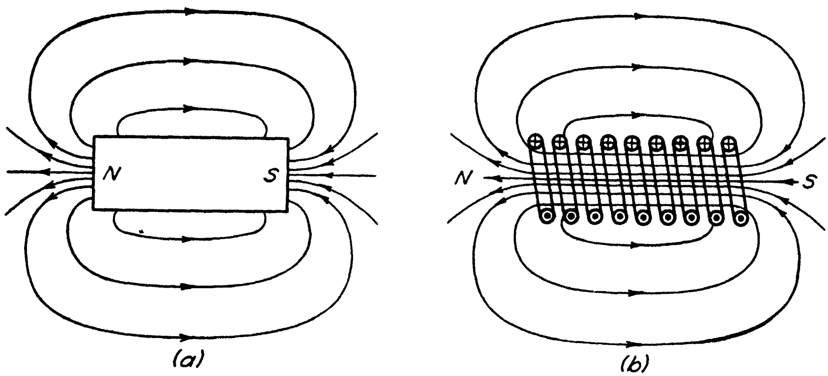


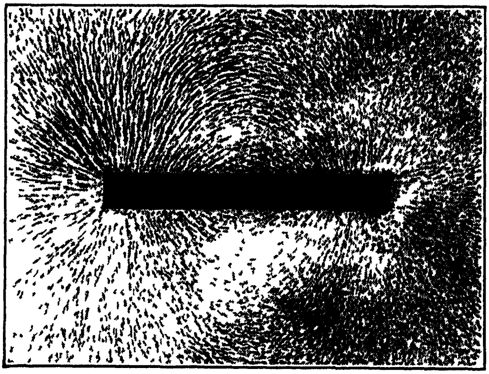
FIG. 1-4. Magnetic fields: (a) field surrounding a bar magnet; (b) field surrounding a coil carrying current.

Figure 2-4 shows two such field plots, taken for the magnet and coil in Fig. 1-4. An inspection of the figures shows that the lines of flux leave the surfaces at an angle of 90°, a condition of basic importance in the study and plotting of magnetic fields. Field plotting, which is one branch of electrical design, confines itself to the making of this type of picture for the complicated magnetic structures of electrical apparatus.

There is a concentration of the lines forming the magnetic field at the ends of the solenoid and bar magnet. This concentration point on either end of the solenoid or the magnet has been designated as the *pole*. The end which is north-seeking is called the north pole and the other the south pole. When the field surrounds a natural or a bar magnet, it is a magnetic field; but when surrounding a coil or a wire it is designated as an electromagnetic field. No matter which source establishes the magnetic field, the laws governing it are the same.

4. Unit Pole. The system of units used in measuring the magnetic field is based on a hypothetical unit pole. Such a pole cannot exist, because the poles always appear in pairs and the individual parts of

any bar magnet will show (if it is divided) the two-pole characteristic of the original magnet. This characteristic, first dealt with by Weber and later expanded by Ewing, explains the phenomenon of magnetism by means of the theory that each molecule is a small magnet with north and south poles. In unmagnetized material these molecular magnets



(a)

(b)

FIG. 2-4. Magnetic field determined by using iron filings: (a) field surrounding a bar magnet; (b) field surrounding a coil carrying current.

are in local groups so arranged that all the poles are neutralized but, in magnetized materials, they align with the others, so that the molecular magnets are not neutralized within the material. If a magnet has true poles, a division of the magnet merely separates the whole into two groups, each with its independent poles. This division may be carried down to the molecule without interrupting the natural polarity. Figure 3-4 shows this in a diagrammatic form.

The unit pole will not have north- and south-pole characteristics but will be an isolated pole of such strength that, when two unit poles of like sign are placed unit distance apart in a vacuum, they repel each other

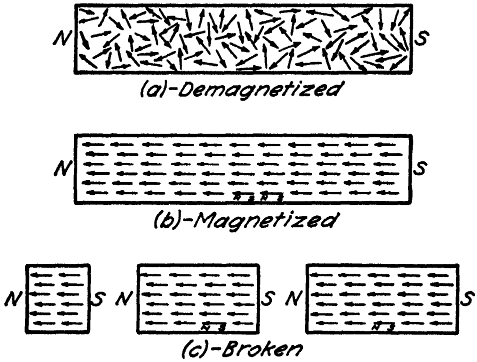


FIG. 3-4. Weber's theory of magnetism, showing the material without magnetism, with magnetism, and the resultant magnets when the magnet is broken.

with unit force (cgs system of units). With such a unit as a base, it follows that magnetic fields and magnetic circuits present one of the difficult phases of electrical engineering.

5. Formation of a Magnetic Field by Electric Current. The presence of a magnetic field surrounding a conductor carrying electric current may easily be proved by the use of cardboard and iron filings with a conductor carrying current. The direction of the lines of a magnetic

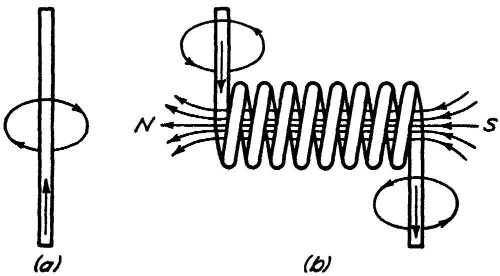


FIG. 4-4. Electromagnet: (a) field surrounding a straight wire; (b) field of a solenoid.

field is conventional; in the external field these lines go from the north to the south pole, and, when a wire carrying current is the source, the lines are in the direction designated by use of the right-hand rule given in Chapter 2. Figure 4-4 shows that when a wire is formed into a coil (solenoid) the lines around the conductor turns are additive and they

form a magnetic system with definite poles. To simulate the magnetic condition of a bar magnet more completely, it is only necessary to thrust a piece of soft iron into a solenoid and immediately the field will concentrate at the ends of the metal. This latter type of magnet is, in various forms, the basis of all electrical generators and electromagnetic machines. Iron improves the field to such an extent that by its use electrical apparatus can be built with economy of both material and space.

6. Inverse Square Law. This law, which has a very wide application in general classical physics, is also applicable to magnetic systems. The force exerted by one pole upon another is directly proportional to the strength of the poles and inversely proportional to the square of the distance between the two poles. This may be expressed by

$$f = k\frac{mm'}{r^2} = \frac{mm'}{\mu r^2}$$

where f is the force in dynes

m and m' are the pole strengths measured in terms of unit pole strength

r is the distance in centimeters

k is the proportionality constant and equal to $1/\mu$ when the permeability of free space equals 1.

Example b. What is the force in dynes exerted between two poles of 1000 and 500 units strength if they are 3 in. apart?

$$f = \frac{1000 \times 500}{(3 \times 2.54)^2}$$

$$f = 8611 \text{ dynes}$$

Experimental research shows that like poles repel and unlike poles attract; therefore, the above expression may be either a force of repulsion or of attraction. This law is confined in its application to point poles and, therefore, to the engineer, its importance lies in a statement of the law and the influence of its various factors on resultant phenomena.

The values m and m' are not expressed in units, but are merely called units of pole strength. The strength of any unknown pole is obtained by measuring the force on a unit pole according to the above law. Though these relationships find importance in definitions and derived magnetic and electromagnetic expressions, in machine design other means are used for determining magnitudes of magnetic fields.

7. Measuring the Magnetic Field. As has been stated, interest in the magnetic field is not centered at the pole, but in and about the magnet

or electromagnet. It is the measure of the influence and magnitude of the field that interests the designer in the application of the magnetic fields to apparatus, because the influence and magnitude of the field determine the induced voltage and the effect on the flow of current in the electrical circuit.

The density of the field (flux density) is the number of lines per square centimeter (or other unit area), measured in a plane perpendicular to the lines. This term is descriptive in itself since it is of the same order as "inhabitants per square mile" or "strands per square inch in a cable." By merely distributing the lines in proper proportion, a diagram may be drawn to represent the true field density. There are two terms which are in common use in designating these lines: namely, *lines of force* and *lines of induction*; and there is considerable discussion of the proper nomenclature at each part of the circuit. It is common practice to designate that part of the magnetic field which may be subjected to physical measure as lines of force and that which cannot be measured by physical means as lines of induction. The original form (lines of force) was the natural outcome of the methods used in plotting magnetic fields, in that the direction in which the magnet pointed indicated the direction of the field. Interchanging these terms in practice makes very little difference, for the wording easily determines the part of the field considered. The particular practice that must be carefully followed is that the conventional designation of the direction of lines in a magnetic field is from *north to south* in the external field, whereas inside the magnet or the magnetic field the designation of direction is from *south to north.* A change of viewpoint will often lead to confusion if this is not kept in mind.

In air or free space, flux density and field intensity are numerically the same, though measured by different units. *Density* measures the *number of lines* per unit area; *intensity* measures the *force* that would be exerted on a unit pole by the field at the point under consideration. This difference is important in understanding the operation of the magnetic circuit and the function of the coefficient of permeability (which measures the relationship between magnetic induction and magnetizing force). Permeability will be taken up in the study of the magnetic circuit.

8. Fundamental Terms of the Magnetic System. Figure 5–4 shows the unit pole placed in free space and the relationship which the fundamental concepts of the magnetic circuit bear to this unit pole. Parallel to this figure will be carried a numerical demonstration to coordinate the significance of the various magnetic terms. When a unit positive pole is placed in free space at a distance of 1 centimeter from a similar

unit pole, each will be repelled by a force of 1 dyne. Since this same force is exerted in three dimensions around the unit pole, there must be one unit of density, that is, one line of force per square centimeter 1 centimeter distant from the pole in any direction. Since a sphere with a radius of 1 centimeter has a surface area of 4π square centimeters, a unit pole will have a total field strength of 4π lines of force.

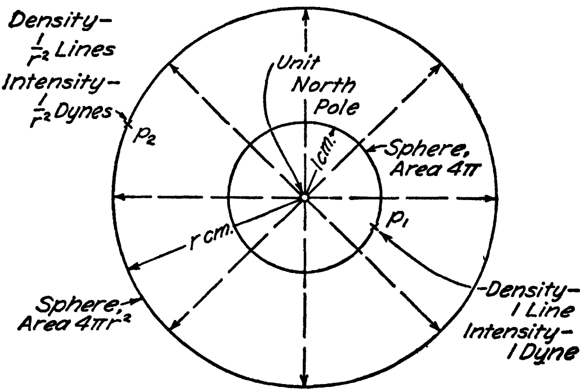


FIG. 5-4. The relationship between the field strength (intensity) and compactness (density) and the unit pole.

If the test pole is moved to a distance of 2 centimeters from the unit pole, the force exerted on the poles will be 0.25 dyne,

$$f = \frac{mm'}{\mu r^2}$$

$$= \frac{1 \times 1}{1 \times 2^2} = 0.25 \text{ dyne}$$

where both m and m' are of unit value. The area of a sphere which is 2 centimeters in radius is 16π square centimeters, but, since the lines given off by the unit pole have not changed, there is a flux density of 0.25 line per square centimeter at 2 centimeters from the unit pole. The intensity follows the inverse square law, and the density is numerically the same as the intensity when the measurement is taken in free space. Magnetic field density and intensity may be defined in simple form by:

Unit flux density (the gauss) is the flux concentration which will give 1 line of force per square centimeter, measured perpendicular to the magnetic lines.

Unit field intensity is the field strength which will exert a force of 1 dyne on a unit pole placed in the field.

9. Symbols and Expressions for Relationships in a Magnetic Field.

$\mathcal{H}$ = field intensity
m = pole strength
$\mathcal{B}$ = flux density
ϕ = total flux
$\mathcal{F}$ = magnetomotive force
A = area
f = force
r = distance
μ = permeability
$\mathcal{R}$ = reluctance
$\mathcal{P}$ = permeance

The following expressions hold only when the poles are point poles and when the test pole does not disturb the existing magnetic condition.

$$f = \frac{mm'}{\mu r^2}$$

$$f = m\mathcal{H}$$

The term flux designates the sum total of the magnetic lines considered in the magnetic circuit or field under discussion, and is normally expressed in "lines" or maxwells.

$$\mathcal{F} = 0.4\pi NI \qquad \phi = \frac{\mathcal{F}}{\mathcal{R}}$$

$$\phi = \mathcal{B}A \qquad \frac{\mathcal{B}}{\mathcal{H}} = \mu$$

10. Magnetic Units. Up to this point, consideration has been given only to the magnetic field and its functioning. Of more importance is the interrelationship of the electric current and the magnetic field. In Fig. 4–4, the flux and the current are linked and it is, therefore, impossible to remove a magnetic line or the current without cutting one by the other. The product of the flux and the turns through which the current flows to produce the flux is called the flux linkage ($N\phi$ or λ). It has already been stated that, when the linear characteristics of the magnetic system are constant, the number of these linkages per unit current is the measure of the inductance. Linkages play an important part in the characteristics of most electrical apparatus. Figure 8–4*b* (p. 72) is essentially a simple transformer with both the primary and secondary windings linked by the same magnetic flux.

The product of the turns and the current is called the ampere-turns (NI). In the electromagnetic system, the ampere-turns are the *cause.* The magnetomotive force is measured by the unit called the gilbert and is proportional to the ampere-turns. Normally, the magnetomotive force is measured in ampere-turns and has more significance to the designer in that form. The magnetomotive force is defined as the work

required to move a unit magnetic pole once around the magnetic circuit. Since the unit pole has 4π lines of flux and the pole flux cuts the turns N of the coil, as it moves around the magnetic circuit, the linkages will be $N\phi$. The work is $N\phi I$; therefore,

$$\mathfrak{F} = W = \frac{N\phi I}{10} = \frac{\lambda I}{10} \quad \text{(practical units)}$$

and

$$\text{total } \phi = 4\pi$$

then

$$\mathfrak{F} = 0.4\pi NI \text{ gilberts}$$

In the magnetic system, as in the electrical circuit, there is an opposition or reaction factor, known as the reluctance (expressed as units of reluctance). When the reluctance appears in calculations as a reciprocal, it is frequently replaced by a term called the permeance ($\mathcal{P}$) which bears the same relationship to the reluctance that admittance bears to impedance ($Y = 1/Z$; see Chapter 5).

Permeability (μ), which was mentioned under magnetic fields, is defined as that property of an isotropic medium which determines, under specific conditions, the magnitude relation between the magnetic induction and the magnetizing force in the medium. Under specific conditions, permeability is measured as the ratio of the magnetic induction to the magnetic force ($\mu = \mathcal{B}/\mathcal{H}$). Permeability is a property of the medium and has dimensions; it is more than a number. The permeability of free space is considered unity and, with the exception of the magnetic materials, it is the practice for engineers in the power field to consider it unity for other materials as well.

11. Laws of the Magnetic Circuit. There are two laws of the magnetic circuit which are of prime importance in power considerations. The first is equivalent to the statement that normal flow follows the path of least opposition; the second reaffirms the general law of nature in establishing a balance between cause and effect, with counteraction the balancing factor.

Mitchell (1750) stated that a magnetic field always tends to conform itself so that the flux attains a maximum value. A piece of iron placed in a magnetic field will lower the reluctance and concentrate the flux. This same condition permits magnetic shielding, where the body to be shielded is enclosed in an iron container and the flux is concentrated in the shielding as a path of lower reluctance.

Rowland (1873) expressed what is often known as Ohm's Law of the Magnetic Circuit: that magnetic flux is directly proportional to the magnetomotive force and inversely proportional to the magnetic reluc-

tance. This law permits the same treatment of a magnetic system as Ohm's Law does of an electrical system, but is less accurate because magnetic flux cannot be confined to a definite path.

When it is assumed that the permeability of the material is independent of the flux (or that variations in current will cause linear variation in flux) the magnetic system is treated much the same as an electric circuit. Whereas electric current may be confined to definite paths, it is impossible to confine flux to one path and an allowance must always be made for the amount of flux that will leak out of the prescribed magnetic path. In configuration, the magnetic paths are very short and have large cross-sections, but the electrical systems have small cross-sections and great length. In determining the reluctance or the counteraction factor of a magnetic system, much depends upon previous history of the particular magnetic material and the state of magnetization. Since the permeability varies with flux density and the two factors (flux and reluctance) are interdependent, the solution of magnetic circuit problems is accomplished either by the use of experimental curves or by the trial and error method.

12. Reluctance. This property of the magnetic circuit opposes the establishment of flux and is expressed by

$$\mathcal{R} = \frac{l}{\mu A}$$

where $\mathcal{R}$ is the reluctance, l is the length of the path in centimeters, A is the cross-sectional area in square centimeters, and μ is the permeability or factor of medium property. Reluctance is directly proportional to the length and inversely proportional to the cross-sectional area. Figure 6–4 shows the relationship between the flux and the permeability and also indicates the complications resulting from an attempt to determine the amount of flux that will flow under specific exciting conditions.

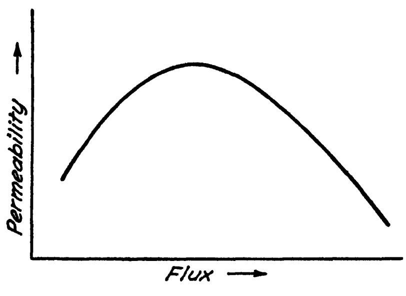


FIG. 6–4. The effect of flux change on the permeability of magnetic material.

Example c. Determine the reluctance of an iron ring if the cross-sectional area is 0.75 sq in., the mean radius is 4 in., and the permeability is 300.

$$\mathcal{R} = \frac{2\pi 4 \times 2.54}{300 \times 0.75 \times 2.54^2}$$
$$= 0.044 \text{ unit}$$

Reluctance bears to the magnetic circuit a relationship similar to that which resistance bears to the electrical system. The two are analogous except in power consumption, for, whereas the resistance consumes power and converts it into heat, the reluctance does not absorb energy. Both are directly proportional to length and inversely proportional to cross-sectional area and both depend in magnitude, upon the material used in the system.

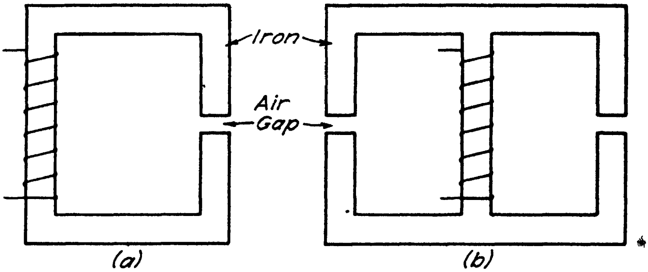


FIG. 7-4. Reluctance in series (a) and parallel (b).

The reluctance may be treated similarly to the resistance when the paths are in series (Fig. 7-4a) or in parallel (Fig. 7-4b).

$$\text{Series } \mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 \cdots \text{etc.}$$

$$\text{Parallel } \frac{1}{\mathcal{R}} = \frac{1}{\mathcal{R}_1} + \frac{1}{\mathcal{R}_2} + \frac{1}{\mathcal{R}_3} \cdots \text{etc.}$$

or

$$\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 \cdots \text{etc.}$$

where $\mathcal{P}$ is the *permeance*.

Example d. In a magnetic circuit 18 in. long are 9 in. of steel having a permeability of 1000 and 9 in. of cast iron having a permeability of 400. Determine the reluctance of the magnetic path if the cross-sectional area is 1 sq in.

$$\begin{aligned} \mathcal{R} &= \frac{9 \times 2.54}{1000 \times 1 \times 2.54^2} + \frac{9 \times 2.54}{400 \times 1 \times 2.54^2} \\ &= 0.0035 + 0.0089 \\ &= 0.0124 \text{ unit} \end{aligned}$$

Example e. What would be the reluctance of the magnetic path if the two pieces of magnetic material, in example *d*, were placed in parallel, making the path 9 in. in length and the cross-sectional area 2 sq in.?

$$\begin{aligned} \mathcal{R} &= \frac{\mathcal{R}_1\mathcal{R}_2}{\mathcal{R}_1 + \mathcal{R}_2} \\ &= \frac{0.0035 \times 0.0089}{0.0124} \\ &= 0.0025 \text{ unit} \end{aligned}$$

More detailed treatment of magnetic circuits and their calculations will be included in the discussions of machines and magnets. The material studied under the magnetic circuit and the magnetic field prepares the way for a comprehensive study of the electrical counteraction known as inductance.

INDUCTANCE

13. Voltage of Self-Induction. If a coil in an electrical circuit is carrying a current, there will be around the coil, or solenoid, a magnetic field which will remain constant as long as the current is constant, but which will change if the current changes. If the current is interrupted, the magnetic field will collapse in such a manner that the lines cut the coil turns (this may be called a collapse of the flux linkages), and, according to the Law of Magnetic Induction, there will be an induced electromotive force in the coil, since there has been a change of flux linkages.

Since the current caused by the induced voltage tends to oppose the change of the flux linkages, the induced electromotive force will be opposite to the direction of change of the impressed voltage. The two foregoing laws were stated and explained in detail in Chapter 2. Application of these laws to the fundamental characteristics of the magnetic system formulates the relationship between the inductance and the physical dimensions of a coil.

The induced electromotive force in a coil will be equal to

$$e = -N\frac{d\phi}{dt} \times 10^{-8} \quad \text{or} \quad -\frac{d(N)\phi}{dt} \times 10^{-8}$$

where the negative sign indicates the opposition to the change of applied voltage and signifies that the applied voltage is considered a reference. If the knowledge of the magnetic circuit is applied to this expression (A and l in square centimeters and centimeters and all other values in cgs units),

$$\mathcal{R} = \frac{l}{\mu A}$$

$$\phi = \frac{\mathcal{F}}{\mathcal{R}} = \frac{\mu A \mathcal{F}}{l}$$

$$\mathcal{F} = 0.4\pi NI$$

$$\phi = \frac{0.4\pi N\mu AI}{l}$$

$$\frac{d\phi}{dt} = \left(\frac{0.4\pi N\mu A}{l}\right)\frac{di}{dt}$$

$$e = -N\frac{d\phi}{dt} \times 10^{-8}$$

$$e = -\left(\frac{0.4\pi N^2 \mu A}{l} \times 10^{-8}\right)\frac{di}{dt}$$

the induced electromotive force takes the form

$$e = -\mathcal{L}\frac{di}{dt}$$

where $\mathcal{L}$ is the coefficient of self-inductance.

The same result will be obtained if the definition

$$\left(\frac{0.4\pi N^2 \mu A}{l} \times 10^{-8}\right)$$

is approached from the linkages per unit current. The inductance so defined is

$$\mathcal{L} = \frac{N\phi}{I} \times 10^{-8}$$

as above

$$\phi = \frac{0.4\pi N \mu A I}{l}$$

and

$$\mathcal{L} = \frac{0.4\pi N^2 \mu A}{l} \times 10^{-8} \text{ (in cgs units)}$$

Example f. What is the electromotive force induced in a system having an inductance of 0.001 henry, when the time is 1 sec and the current is expressed by the equation $i = 10 \sin 377t$?

$$\begin{aligned} e &= -0.001\frac{d}{dt}(10 \sin 377t) \\ &= -0.001 \times 3770 \cos (360 \times 60 \times 1)^\circ \\ &= -0.001 \times 3770 \times 1 \\ &= -3.77 \text{ volts} \end{aligned}$$

Example g. Calculate the inductance of a solenoid 12 in. long made up of 1000 turns of wire on a tube having a cross-sectional area of 1 sq in. The tube is non-magnetic and leakage flux will be neglected.

$$\begin{aligned} \mathcal{L} &= \frac{4\pi(10^3)^2 \times (1 \times 2.54^2) \times 1}{12 \times 2.54 \times 10^9} \\ &= 0.00266 \text{ henry} \end{aligned}$$

These expressions for inductance point directly to the values of the dimensions and the material characteristics influencing its magnitude. The turns (N) will be the controlling factor since their influence follows the square law. Other factors as well as the turns are controlled in design and, as a rule, inductance design is based on empirical curves and formulas.

14. Mutual Inductance. It has already been pointed out that the change of the flux linkages in a closed electrical circuit will induce an electromotive force. It is equally true that the change of linkages involving a second coil or circuit will induce an electromotive force in

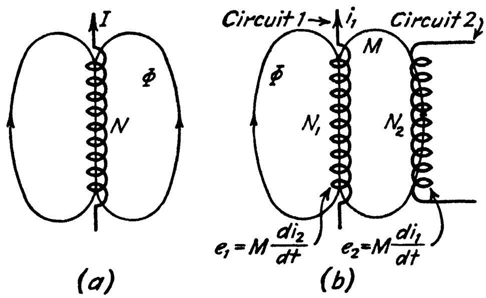


FIG. 8–4. Self-inductance (a) and mutual inductance (b).

that circuit. Figure 8–4 shows both a simple coil (not linked magnetically with any other circuit) and a circuit carrying current (which is linked magnetically with another coil). If the linkages in the first coil change (Fig. 8–4b), the lines of force will cut the second coil and set up in it an induced electromotive force. If this results, the two coils have mutual inductance.

Mutual inductance is the common property of two associated electrical circuits which determines, for a given rate of change of current in one of the circuits, the electromotive force induced in the other. Thus,

$$e_1 = M\frac{di_2}{dt}$$

$$e_2 = M\frac{di_1}{dt}$$

where e_1 and i_1 are in circuit 1, e_2 and i_2 are in circuit 2, and M is the coefficient of mutual inductance. This type of inductance is very important in some electrical machines, one of which is the transformer.

If the coils that are coupled are wound on top of one another and the flux leakage is neglected, the mutual inductance is

$$M = \frac{0.4\pi N_1 N_2 \mu A \times 10^{-8}}{l}$$

where μ is the permeability of the core, A is the cross-sectional area, and l is the length (cgs units) of the magnetic path. N_1 and N_2 apply to the coils as indicated in Fig. 8–4*b*.

Example h. Neglecting the leakage flux, what is the mutual inductance between two transformer coils of 1000 and 100 turns, respectively, if the coils are 10 in. long, 3 in. in diameter, and wound on bakelite forms?

$$M = \frac{4\pi(1000)(100) \times (1.5)^2\pi \times (2.54)^2 \times 1}{10 \times 2.54 \times 10^9}$$

$$= 225.6 \times 10^{-5} \text{ henrys}$$

It can be shown that the mutual inductance may be expressed in terms of the individual inductances of the two systems. However, a factor must be introduced to account for the leakage, which (though it is sometimes negligible) always exists when two systems are magnetically coupled. This leakage factor is known as the coefficient of coupling. Expressing M in these terms, the value of M is

$$M = K\sqrt{\mathcal{L}_1 \mathcal{L}_2}$$

where K is the coupling coefficient (leakage factor) and $\mathcal{L}_1$ and $\mathcal{L}_2$ are the respective inductance coefficients of the two systems.

Example i. What is the coefficient of coupling in the foregoing problem?

$$\mathcal{L}_1 = \frac{4\pi(1.5^2\pi \times 2.54^2)}{10 \times 2.54 \times 10^9} \times (10^3)^2$$

$$\mathcal{L}_2 = \frac{4\pi(1.5^2\pi \times 2.54^2)}{10 \times 2.54 \times 10^9} \times (10^2)^2$$

$$\mathcal{L}_1 = 225.6 \times 10^{-4}$$

$$\mathcal{L}_2 = 225.6 \times 10^{-6}$$

$$M = 225.6 \times 10^{-5}$$

$$K = \frac{225.6 \times 10^{-5}}{\sqrt{(225.6 \times 10^{-4})(225.6 \times 10^{-6})}} = 1$$

The coefficient of coupling is unity, since the leakage flux has been neglected. In practice the coefficient is actually less than unity.

THE ELECTRICAL CIRCUIT AND INDUCTANCE

In these studies, only steady-state current flow will be considered. Periods of switching and change of load produce transient effects and the duration of each is only a small fraction of a second. These periods are of utmost importance in the operation of apparatus and of electrical systems, because they determine the possible interconnections and stabilities of operation. This field is extensive and very complex; it deals with exponential conditions superimposed upon steady-state operation. The study of transients is desirable for advanced electrical engineering and graduate students. The material in this book is concerned, in general, with conditions in which all transients have disappeared and the system is operating under a steady-state condition, which permits direct application of mathematics without determination of constants of integration or solutions of differential equations.

15. Inductance in the A-C System. Here, steady state must first be established and all transients eliminated so that observations will indicate that the voltage is following a sinusoidal wave. In this event, instead of dealing with steady flow, the problem is one of wave motion and is comparable to studies of harmonic motion. In dealing with the electrical field, all phenomena fall into one of two classes: either *unidirectional* flow or *wave* motion. The greater portion of the student's work in mechanics and engineering is concerned, in general, with the uniflow problem but, in electrical engineering, the greater interest lies in the wave motion or reciprocating field.

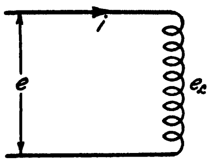


FIG. 9-4. A pure inductance forming an electrical circuit.

Figure 9-4 shows a pure inductance, in which the resistance of the wire is (in theory) non-existent or (in practice) negligible. This inductance, if subjected to the voltage e supplied by an a-c generator, will permit a current i to flow in the system. What will be the effect of the inductance and to what degree will it counteract the flow of current? What will be the resultant effect if the *cause* confronts a *counteraction*, such as an inductance, in an a-c system?

When the system has reached a steady state with sine wave electromotive force, the induced electromotive force in the coil will be

$$e_\mathcal{L} = -\mathcal{L}\frac{di}{dt}$$

Therefore the impressed voltage is

$$e = \mathfrak{L} \frac{di}{dt}$$

where

$$e = E_m \sin \omega t$$

Substituting

$$E_m \sin \omega t = \mathfrak{L} \frac{di}{dt}$$

$$i = \frac{E_m}{\mathfrak{L}} \int \sin \omega t \, dt$$

$$= \frac{E_m}{\omega \mathfrak{L}} (-\cos \omega t)$$

$$= \frac{E_m}{\omega \mathfrak{L}} \sin \left(\omega t - \frac{\pi}{2}\right)$$

$$= \frac{E_m}{2\pi f \mathfrak{L}} \sin (360ft - 90)^\circ$$

the instantaneous current will be a maximum when the value of sin $[\omega t - (\pi/2)]$ is equal to one, which makes

$$I_m = \frac{E_m}{\omega \mathfrak{L}} = \frac{E_m}{2\pi f \mathfrak{L}}$$

When this is substituted in the above expression for instantaneous current, the expression takes the form

$$i = I_m \sin \left(\omega t - \frac{\pi}{2}\right)$$

In the simple system shown, the following conditions are effective when sine wave voltage is applied to a pure inductance and the voltage and current are

$$e = E_m \sin \omega t$$

$$i = I_m \sin \left(\omega t - \frac{\pi}{2}\right)$$

respectively. Sinusoidal voltage gives rise to sinusoidal current, which is another sine wave 90° displaced from the original voltage and passing through zero on the time axis 90 electrical degrees later.

Example j. If a sinusoidal voltage with an amplitude of 100 is impressed on an inductance of 0.01 henry, what will be the amplitude of the current wave if the frequency of the system is 60 cycles?

$$I_m = \frac{100}{2\pi \times 60 \times 0.01}$$

$$= 26.53 \text{ amp}$$

Example k. A sinusoidal voltage, $e = 100 \sin 377t$, is impressed on a pure inductive reactance of 10 ohms. What will be the instantaneous current at 0.1 sec after the instantaneous voltage, which is increasing in magnitude, passes through zero value?

$$i = I_m \sin\left(\omega t - \frac{\pi}{2}\right)$$

$$I_m = \tfrac{100}{10} = 10$$

$$i = 10 \sin\left(377t - \frac{\pi}{2}\right)$$

$$i = 10 \sin\,(360 \times 60 \times 0.1 - 90)^\circ$$

$$i = -10 \text{ amp}$$

Since the sine wave is a harmonic motion, it may be represented by a rotating vector. A complete diagram of the vectors and the sine waves appears in Fig. 10–4. It is well to study the analytic geometrical rela-

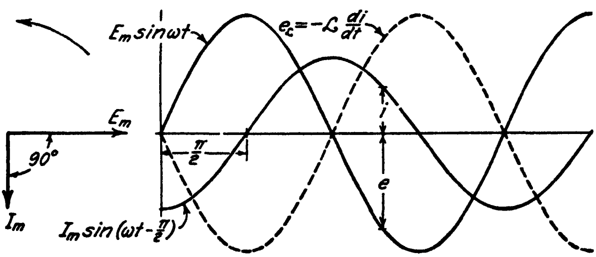


FIG. 10–4. The relationship of the instantaneous current and voltage in a pure inductive circuit, with impressed voltage as a reference.

tionships of the waves as well as the vector representation, because both are used in showing the phenomena present in all alternating systems and apparatus.

16. Voltage-Current Relationships in the Alternating System Containing Pure Inductance. It has been shown that the maximum current is expressed by

$$I_m = \frac{E_m}{\omega \mathcal{L}} = \frac{E_m}{2\pi f \mathcal{L}}$$

The terms ω and $\mathcal{L}$ occur in all considerations of this relationship and only in rare cases are they separated, so that it is common practice to replace them by a single term $X_{\mathcal{L}}$, called the inductive reactance. This term, when expressed by the ratio of voltage to current, takes the form

$$X_{\mathcal{L}} = \frac{E_m}{I_m}$$

Inductive reactance may be called the proportionality factor between the cause and the effect.

Example l. What will be the value of the inductive reactance in a 60-cycle electrical system when the instantaneous voltage and current are expressed by $e = 100 \sin 377t$ and $i = 10 \sin [377t - (\pi/2)]$, respectively?

$$X_{\mathcal{L}} = \tfrac{100}{10} = 10 \text{ ohms}$$

17. Inductive Reactance. Inductive reactance is the opposition of a portion of a circuit either to a sinusoidal current of the same frequency as the voltage or to any component of a periodically varying current. If there is no other source of power in the portion of the circuit under consideration, the reactance is equal to the ratio of the quadrature (90°) component of the potential difference for the particular frequency to the value of the current for that frequency. Vectorially, the quadrature component of the voltage is the component of voltage at 90° to the current in time relationship.

The inductive reactance is expressed by

$$X_{\mathcal{L}} = \omega\mathcal{L}$$

where ω is the angular velocity, or number of radians turned per second, and, if the frequency f is fixed, it will be $2\pi f$; that is, 2π radians will be covered per cycle and f cycles will be passed through per second. When this is substituted in the above form, the inductive reactance becomes

$$X_{\mathcal{L}} = 2\pi f\mathcal{L}$$

Replacing the value of $\mathcal{L}$,

$$X_{\mathcal{L}} = 2\pi f \frac{0.4\pi N^2 \mu A}{l} \times 10^{-8}$$

This expresses the inductive reactance in terms of the physical properties of the system. The configuration and the frequency of the system determine the value of the opposition which is caused by inductive reactance.

This property of the electrical system (counteraction) is measured in the common unit of opposition, the ohm, which is the unit used to measure the magnitude of resistance.

Example m. If a solenoid 12 in. long with an air core 1 sq in. in cross-sectional area and having 1000 turns is placed on a 25-cycle system, what will be the resultant inductive reactance of the coil?

$$X_{\mathcal{L}} = \frac{2\pi \times 25 \times 4\pi(10^3)^2 \times 1 \times (2.54)^2 \times 1}{12 \times 2.54 \times 10^9}$$

$$= 0.4178 \text{ ohm}$$

Summary. There have been rather definite restrictions placed upon the conditions set forth in the study of the effect of inductance; in order to emphasize these restrictions, the following summary is given.

1. The wave shape is sinusoidal.
2. The frequency is constant.
3. The permeability is constant and independent of current changes.
4. The system has reached a steady state of operation (transients are not considered).

Since complicated periodic waves may be broken down into a series of sine waves, the above conditions may be extended beyond the simple sine wave in advanced studies.

18. Inductive Reactance in Series and Parallel. If inductive reactances are connected in series the same current passes through each reactance, Fig. 11–4. Assuming sinusoidal current, $i = I_m \sin \omega t$; the voltage across each reactance will be

$$e_1 = x_1 I_m \sin \omega t$$

$$e_2 = x_2 I_m \sin \omega t$$

$$e_3 = x_3 I_m \sin \omega t$$

and the total voltage will be

$$\begin{aligned} e &= e_1 + e_2 + e_3 \\ &= x_1 I_m \sin \omega t + x_2 I_m \sin \omega t + x_3 I_m \sin \omega t \\ &= (x_1 + x_2 + x_3)\, I_m \sin \omega t \\ &= x_0 I_m \sin \omega t \end{aligned}$$

$$\omega\mathcal{L}_0 = x_0 = x_1 + x_2 + x_3 = (\omega\mathcal{L}_1 + \omega\mathcal{L}_2 + \omega\mathcal{L}_3) = \omega(\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3)$$

$$\mathcal{L}_0 = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$$

Therefore, in a series circuit the total inductive reactance or inductance will be the sum of the individual parameters of the circuit.

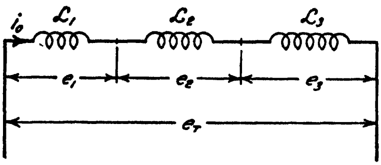


FIG. 11–4. Inductances in series.

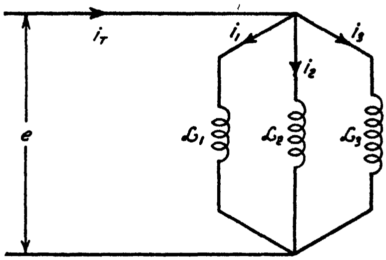


FIG. 12–4. Inductances in parallel.

If inductive reactances are connected in parallel, Fig. 12–4, the voltage across each inductive reactance is the same; $e = E_m \sin \omega t$. With a sinusoidal voltage across each branch the current flow through each inductive reactance will be

$$i_1 = \frac{E_m \sin \omega t}{x_1}$$

$$i_2 = \frac{E_m \sin \omega t}{x_2}$$

$$i_3 = \frac{E_m \sin \omega t}{x_3}$$

and the total current will be

$$\begin{aligned} i &= i_1 + i_2 + i_3 \\ &= \frac{E_m \sin \omega t}{x_1} + \frac{E_m \sin \omega t}{x_2} + \frac{E_m \sin \omega t}{x_3} \\ &= \left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right) E_m \sin \omega t \\ &= \frac{E_m \sin \omega t}{x_0} \end{aligned}$$

$$\frac{1}{\omega \mathcal{L}_0} = \frac{1}{x_0} = \left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right) = \frac{1}{\omega \mathcal{L}_1} + \frac{1}{\omega \mathcal{L}_2} + \frac{1}{\omega \mathcal{L}_3}$$

$$\frac{1}{\mathcal{L}_0} = \frac{1}{\mathcal{L}_1} + \frac{1}{\mathcal{L}_2} + \frac{1}{\mathcal{L}_3}$$

Therefore, in a parallel system containing only inductive reactance or inductance, the reciprocal of the total system parameter will be the sum of the reciprocals of the individual parameters constituting the parallel circuits.

19. A Second Approach to Current-Voltage Relationship in an A-C System Containing Inductance. Assume, in this instance, that it is desired to determine the nature of the voltage required to send a *sinusoidal current* through an inductance. Returning to the expression for induced voltage,

$$e_{\mathcal{L}} = -\mathcal{L}\frac{di}{dt}$$

Therefore, the impressed voltage will be

$$e = \mathcal{L}\frac{di}{dt}$$

and with the sine wave current expressed by

$$i = I_m \sin \omega t$$

then

$$e = \mathcal{L}\frac{d}{dt}(I_m \sin \omega t)$$

Differentiating and simplifying, this takes the form

$$e = I_m\omega\mathcal{L} \cos \omega t$$

$$e = I_m\omega\mathcal{L} \sin\left(\omega t + \frac{\pi}{2}\right)$$

When the sine of the angle is unity e is a maximum in value so that

$$E_m = I_m\omega\mathcal{L} = I_m X_{\mathcal{L}}$$

Therefore, the expression for the instantaneous impressed voltage on the inductance is

$$e = E_m \sin\left(\omega t + \frac{\pi}{2}\right)$$

and the respective phase relationship between the voltage and current is shown in Fig. 13–4.

This second approach brings out the importance of reference and shows that the question of leading or lagging depends upon selection of the reference. If, as in the first consideration, the voltage is taken as a reference, the current lags the voltage; if, as in the second instance, the current is the reference, the voltage leads the current. These are one

and the same thing, but the choice of reference is often confusing when considering the relationship of voltage and current. In all problems, some reference must be chosen and rigorously adhered to throughout the discussion. If, for any reason, it is necessary to change the reference, every vector or wave must be changed to the new reference to insure correct results.

The phase difference between two sinusoidal quantities which have the same period is the fractional part of a period (not greater than one half) through which the independent variable must be assumed to have

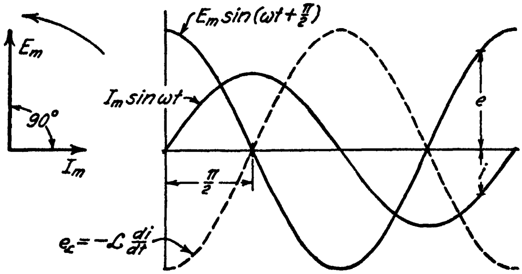


FIG. 13–4. The relationship of the instantaneous current and voltage in a pure inductive circuit, with current as the reference.

advanced with respect to the dependent variable in order that similar values of the fundamental components of the two quantities shall coincide.

If the origin of a sinusoidal quantity is taken at the instant when the independent variable passes through zero increasing in a positive direction, the condition is one of lag or lead, depending upon whether the dependent variable is negative or positive at this instant.

20. Inductance in the D-C System. Returning to Fig. 9–4, used in the previous discussion of the a-c system, apply direct current from either a battery or a d-c generator and note the resultant effect.

The induced electromotive force in an inductance is expressed by

$$e_\mathcal{L} = -\mathcal{L}\frac{di}{dt}$$

When a steady state of direct current has been established, the current is constant in value, that is,

$$I = k$$

Substituting this in the above expression,

$$e_\mathcal{L} = -\mathcal{L}\frac{d}{dt}(k) = 0$$

The opposition voltage is equal to zero and will, therefore, not require any of the impressed voltage to overcome the effect of inductance in the system.

Consider the expression for the current flowing in a circuit containing a pure inductance, where

$$i = \frac{1}{\mathcal{L}} \int e \, dt = \frac{e}{\mathcal{L}} t$$

When the voltage is constant in value and the inductive effect ($\omega\mathcal{L} = \omega N\phi/I$) is zero, the instantaneous current will be infinite in value because the impressed voltage meets no opposition.

It has been shown in the foregoing discussion that in a d-c system, when a steady state has been established, the inductance is without effect and need not be considered.

21. Stored Energy in the Magnetic Field. It has been previously stated that the magnetic field does not consume energy from the electrical system but merely stores energy through one portion of the cycle to return the same energy to the system during another portion of that cycle. This stored energy makes itself evident at the switch contacts when a circuit containing a high inductance is interrupted, and may be dangerous to both life and equipment if not taken into consideration when handling circuits. An example of the release of such an energy source may be witnessed, for, when the field switch on either an alternator or a d-c generator is opened, an arc is formed.

The energy stored in the magnetic field may be obtained by considering the current and voltage of the circuit which produces the magnetic field.

The power in an electric system is

$$p = ei \tag{a-4}$$

The energy is represented by

$$dJ = ei \, dt$$

Substituting the value for e in the inductive system,

$$\begin{aligned} dJ &= \mathcal{L} \frac{di}{dt} i \, dt \\ &= \mathcal{L} i \, di \\ J &= \mathcal{L} \int_{I_0}^{I_t} i \, di \\ &= \tfrac{1}{2} \mathcal{L}(I_t^2 - I_0^2) \\ J &= \mathcal{L} \frac{I^2}{2} \end{aligned}$$

where I_0 equals zero. When $\mathcal{L}$ is in henrys, I is in amperes, J is measured in joules. The amount of energy which is stored in the magnetic field may be expressed in other forms by the substitution of equivalent values in the magnetic expressions with the result that expressions for determining the pull of magnetic armatures and lifting solenoids may be determined.

The energy stored in the magnetic field is similar to that stored in a weight that has been lifted to the top of a pile driver. It requires energy to lift the weight into position but none to keep it in position. The stored energy in the head of the pile driver is returned when the weight is released and the operating cycle is completed.

Example n. Find the energy stored in the magnetic field of a ring solenoid having 1000 turns, a mean radius of 4 in., and a cross-sectional area of 0.75 sq in., if the permeability of the core is 300 and a maximum current of 10 amp flows through the winding.

$$J = \frac{1}{2} \times \frac{4\pi \times (10^3)^2 \times 0.75 \times 2.54^2 \times 300}{2\pi 4 \times 2.54 \times 10^9} \times 10^2 = 14.29 \text{ joules}$$

Example o. Determine the energy stored in the magnetic field of a coil during the part of a cycle which lies between the zero axis and the positive maximum of the current if the inductance of the coil is 10 henrys and the current (which is sinusoidal) is recorded as 10 amp.

$$I_m = 10 \times \sqrt{2} = 14.14 \text{ amp}$$

$$W = 10 \times \frac{(14.14)^2}{2} = 1000 \text{ joules}$$

PROBLEMS

1–4. What effective current value would induce an average voltage of 48 volts in a coil having an inductance of 0.2 henry? The current change follows a sine law from 0 to $\pi/2$ and the system has a 60-cycle characteristic.

2–4. A cast iron core, with a length of 10 in. and a diameter of 2 in., has a reluctance of 0.02 unit. If 10 amp flow through 200 turns of wire wrapped on the core, what will be (*a*) the flux density, (*b*) the inductance, (*c*) the permeability?

3–4. The magnetic path of a 2-pole generator has a reluctance of 0.25 unit. The 6-sq-in. poles are wound with coils containing 50 layers of 100 turns on each pole. What current will produce a flux density of 5×10^4 lines per square inch in the poles?

4–4. A flux of 5×10^5 lines is reduced to zero in 0.1 sec. The flux was established by 25 amp flowing through 5000 turns wound on the core. (*a*) What is the induced voltage? (*b*) What is the inductance of the coil?

5–4. The dimensions of a magnetic system are changed so that the length is reduced to 50 per cent of its original value and the cross-section area to 25 per cent. What will be the ratio of the original to the final inductance?

6–4. A motor field circuit of 2000 turns and carrying 2 amp producing a flux of 10^6 lines is interrupted in 0.1 sec. What is the voltage of self-induction?

7–4. A coil of 1000 turns on a core with an area of 1 sq in. and a length of 12 in. has a voltage $e = 75 \sin [2000\pi t + (\pi/3)]$ applied to its terminals. What will be the instantaneous current at time $t = 0.01$ sec?

8–4. What will be the instantaneous value of the voltage, when the instantaneous current is a maximum, if "e" $= 50 \sin [377t - (\pi/6)]$ is connected to a coil with an inductance of 0.1 henry?

9–4. A coil with a 60-cycle current $i = 10 \sin \omega t$ flowing through its 500 turns has a maximum flux of 2×10^4 lines. Determine the instantaneous expression for voltage across the coil at the end of $1/240$ sec.

10–4. When a 1600-turn coil, with a resistance of 125 ohms, is placed across 100 volts d-c, 8 joules are stored in the magnetic field. What is the flux?

11–4. Two coils have a common magnetic path with self-inductances of 20 henrys each and a mutual inductance of 10 henrys. If the current through the first coil is 2 amp and is reduced to zero in 0.01 sec, determine the voltage in the second coil.

12–4. When a current of 15 amp flows through a coil of 400 turns it produces a field of 5×10^5 maxwells. (*a*) What is the inductance, and (*b*) what will the inductance be if the current is increased to 25 amp?

13–4. (*a*) Determine the inductance of a coil of 800 turns through which 3 amp flows to produce 5.2×10^5 maxwells. (*b*) What is the energy stored in the field?

14–4. A voltage $e = 75.4 \sin [377t + (\pi/6)]$ is impressed on a coil of 100 turns having an inductance of 0.1 henry. Determine (*a*) the current flowing, (*b*) the maximum flux threading the coil, (*c*) the instantaneous current when $t = 0.01$ sec.

15–4. A current $i = 10 \sin \omega t$ flows through an inductance coil with 500 turns and a flux of 2×10^4 maxwells. If the frequency of the system is 60 cycles (*a*) what will be the instantaneous voltage across the inductance at $t = 1/240$ sec, (*b*) what is the inductance of the coil?

CHAPTER 5

CAPACITANCE: REVERSIBLE COUNTERACTION

Chapters 3 and 4 have discussed the counteraction caused by resistance and inductance and have dealt with magnetic fields and with the flow of electric currents in conductors. This phenomenon of magnetic fields produced by electric currents is called electromagnetics and a great number of problems must be solved in this branch of electrical engineering.

Electrical charges or electricity may be stationary. Even though they are stationary, these charges of electricity create fields which in turn produce electrical stresses in the surrounding media. As an example, the insulating requirements for a high voltage transmission line are governed by the stresses in the insulating materials and surrounding media. The division of electrical engineering which deals with these electrical charges and the stresses produced by them is called electrostatics.

1. Law of Electrostatic Attraction—Coulomb's Law. "The force of attraction or repulsion between two charges of electricity concentrated at two points in an isotropic medium is proportional to the product of their magnitudes and is inversely proportional to the square of the distance between them. The force between unlike charges is an attraction; between like charges a repulsion." *

The force is also dependent upon the characteristics of the surrounding medium. This law can be expressed in equation form as

$$f = \frac{q_1 q_2}{kr^2}$$

where f is the force exerted in dynes, q_1 and q_2 are the charges of electricity in electrostatic units, r is the distance between charges in centimeters, and k is a constant depending upon the medium surrounding the charges q_1 and q_2.

The dielectric constant k depends upon the kind of material and can be compared to the constant ρ, the coefficient of resistivity. The values

* From American Standard Definitions of Electrical Terms—1941.

of k may be considered as a comparison between the media and air as a dielectric. For this reason, the value of k for air is considered unity. Table I–5 gives a list of the dielectric constants for the most common media.

TABLE I–5

TABLE OF DIELECTRIC CONSTANTS AND DIELECTRIC STRENGTH [1]

Material	Dielectric Constant k	Dielectric Strength Volts per Mil Thickness
Air	1	76
Cloth (varnished)	3–6	500–1250
Fiber	2	150– 300
Glass	5.5–10.0	150– 300
Mica	2.5– 6.0	1000–2500
Paraffin wax	1.8– 2.2	200– 300
Porcelain	4.0– 6.0	200– 300
Rubber and rubber compounds	2.0– 4.0	250– 500
Transformer oil	2.5	200– 250

[1] For a more complete list consult Volume IV, Pender's Electrical Engineers' Handbook, John Wiley and Sons.

The unit of electrostatic quantity (esu) is called the statcoulomb. It is defined as that quantity of electrical charge which repels, with a force of 1 dyne, a similar charge placed in air at a distance of 1 centimeter. The practical unit of quantity is the coulomb and is equal to 3×10^9 statcoulombs.

Example a. Two charges, $q_1 = 10$ esu and $q_2 = 15$ esu, are 1 cm apart, in air. If the charges are of like sign, determine the magnitude and direction of the force exerted between these charges.

For air, Table I–5, the value of $k = 1$.

$$f = \frac{10 \times 15}{1 \times (1)^2} = 150 \text{ dynes repulsion}$$

Example b. If the dielectric is glass and the two charges of the previous problem are only 0.1 cm apart, what is the magnitude and sign of the force exerted?

From Table I–5, glass has a coefficient from 5.5 to 10.00. For this problem use $k = 8$.

$$f = \frac{10 \times 15}{8 \times (0.1)^2} = 1875 \text{ dynes repulsion}$$

2. Electrostatic (Dielectric) Fields. If electrostatic forces exist in a medium, there is present in that medium an electrostatic field. The direction of the electrostatic force is the direction in which a unit positive charge will move, and its strength or intensity depends upon the force exerted in dynes on the unit charge. Similar to electromagnetic field study, the field strength or intensity can be represented by lines called electrostatic force lines. These lines of electrostatic force will obey the same laws as lines of magnetic force. In air (in which $k = 1$), the number of lines of electrostatic force per square centimeter is equal to the force exerted in dynes on unit charge at that point. If the medium has a dielectric constant k, the number of lines is equal to k times the force exerted on unit charge. The force exerted on unit charge decreases as the dielectric constant is increased for a constant electrostatic field.

The total electrostatic lines radiating from a unit charge is 4π; therefore, the total lines radiating from any charge q is $4\pi q$ lines. These lines of electrostatic force leave or enter a conducting surface at right angles to that surface. The electrostatic field produces a condition of electrostatic stress in the surrounding medium and, if this stress becomes great enough to cause electrical failure of the medium, an electric arc discharge takes place and the medium becomes a conductor of electric current.

3. Dielectrics and Dielectric Strength. The surrounding medium in electrostatic systems is called the electrostatic insulator or dielectric. A perfect dielectric has been defined as one in which all the energy required to establish an electric field in the dielectric is returned to the electric system when the field is removed. A perfect dielectric must have zero conductivity. The strength (electrically) of the dielectric as an insulator in an electric circuit is measured by the amount of potential difference necessary to cause a mechanical failure with resultant arc discharge through the mechanical failure. The dielectric strength also depends upon the method of applying the electrical stress. The ability of a medium or substance to resist dielectric failure is called its dielectric strength.

A medium may have good insulation resistance and still have poor dielectric characteristics. A distinction can be made between insulating properties and dielectric properties in that a good insulating medium permits only very weak (negligible) currents to flow, whereas a medium with a high dielectric strength requires high electrostatic potential difference to cause electrical breakdown and mechanical failure. Insulation resistance is separable into two components: volume resistivity and surface resistivity. The insulation resistance is dependent upon the resistivity of the medium and is expressed in megohms per centimeter; it is measured between parallel surfaces having 1 square centi-

meter of area. The voltage required to cause dielectric failure is expressed in volts per inch thickness or volts per mil (0.001 inch) of dielectric. This voltage per unit thickness of dielectric is the potential or voltage gradient.

If the insulating properties of the two dielectrics, air and rubber, are compared, it is found that the dielectric strength of air is approximately 76 volts per mil thickness, and that of rubber is from 250 volts to 500 volts per mil thickness. Although rubber has the higher dielectric strength, it permits a larger leakage current to flow than does a similar thickness of air, indicating that rubber has poorer insulating properties than air. The leakage current for rubber at low voltages (440 volts and less) is negligible. In determining the performance of a dielectric medium, both its dielectric strength and insulating property must be considered. Table I–5 includes a comparison of the dielectric strengths of the more widely used insulating materials.

4. Capacitance, Condenser. Capacitance (also called capacity) is that property of a system of electric conductors and the dielectric medium between them that permits the storage of electricity when there is potential difference existing between the conductors. If a body is being charged with electricity, its potential is raised; and, when one unit of electricity is required to raise its potential one unit, the body has unit capacity. The potential increase depends directly upon its charge and inversely upon its capacity.

$$e = \frac{q}{C} \quad \text{and} \quad C = \frac{q}{e}$$

where e = increase in voltage
q = increase in charge
C = capacitance.

The practical unit of capacitance is the farad; a body has a capacitance of 1 farad when 1 coulomb of electricity is required to raise its potential 1 volt. The farad as a unit is too large for engineering use and, for that reason, the capacitance of bodies is generally expressed in microfarads (10^{-6} farad).

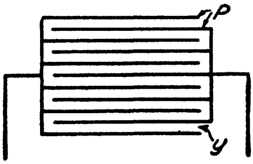


FIG. 1–5. A condenser, consisting of a series of surfaces (P) separated by a dielectric (y).

When conducting surfaces are so arranged that large quantities of electricity can be stored, the name condenser is used for this assemblage. A condenser consists of a group of parallel plates separated by some dielectric medium, as in Fig. 1–5.

In general, it is difficult to calculate the capacitance of a condenser from its physical dimensions, because of its complex construction and

the inability to determine accurately the dielectric constant. The capacitance of a parallel plate condenser as shown in Fig. 1–5 can be closely approximated from the expression below, if it is assumed that the electrostatic lines are always perpendicular to the surface of the plates and that the fringing is negligible.

The capacitance of a condenser can be determined from the expression:

$$C = \frac{8.842kA}{d \times 10^8} \text{ microfarads}$$

where k = dielectric constant
A = total dielectric area, carrying a charge, expressed in square centimeters
d = distance between plates in centimeters.

When the area (A) of the dielectric carrying charge is expressed in square inches and the thickness of the dielectric is expressed in inches, the capacitance of a condenser can be expressed as

$$C = \frac{22.46kA}{d \times 10^8} \text{ microfarads}$$

Condensers may also be made by rolling into a compact form two sheets of lead foil or tin foil separated by some dielectric material (usually oil-treated paper or wax paper) and insulating the leads to the metallic sheets. Many different types of condensers are on the market as a result of the developments in the field of radio. Condensers used on the high voltage systems generally use a special oil as the dielectric.

Example c. Determine the capacity of a 13-plate air condenser if the area of each plate is 3 sq in. and the plates are spaced 0.03 in. apart.

The total charged area is the area of 12 plates or 36 sq in. The dielectric constant is

$$C = \frac{22.46 \times 1 \times 36}{0.03 \times 10^8} = 0.00027 \text{ microfarad}$$

Example d. A condenser consists of two sheets of tin foil and dielectric material of paraffin waxed paper wound together with leads to the foil sheets brought out. The foil sheets are each 25 ft long and 3.75 in. wide. The dielectric material is 0.005 in. thick with a dielectric constant of 2.1. What is the capacitance of this condenser?

$$C = \frac{22.46 \times 2.1 \times 25 \times 12 \times 3.75 \times 2}{0.0005 \times 10^8} = 0.000212 \text{ microfarad}$$

In any condenser, the capacitance depends upon the total area of the insulated surfaces, and its voltage rating depends upon the kind of dielectric used. The capacitance of a condenser can be calculated by de-

termining the quantity of electricity which is required to give unit increase in potential difference between the condenser terminals. These relationships have been stated previously as

$$C = \frac{q}{e} \quad \text{or} \quad q = Ce$$

The quantity of electricity stored in a condenser is directly proportional to its capacity (farads) and the potential difference between terminals. The practical unit of quantity of electricity is the coulomb, which may be defined as the quantity of electricity passing a point in a conductor when 1 ampere of current flows for 1 second, or

$$q = \int i \, dt$$

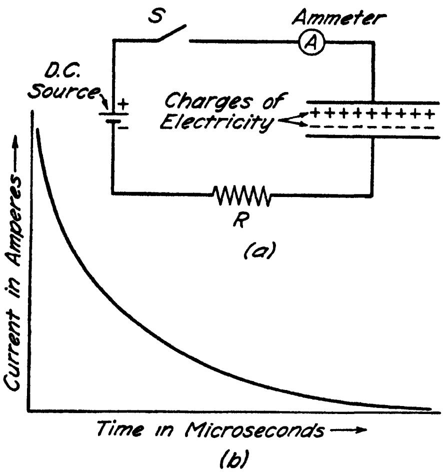


FIG. 2–5. (a) Condenser connected to a d-c source. (b) Variation of current while condenser is being charged.

If a d-c supply is impressed across the terminals of a condenser, the current will flow until the condenser becomes fully charged. The current will then cease to flow, because the potential difference between condenser terminals will be equal in magnitude but opposite in sign to the impressed voltage. Since the potential difference between plates is zero at the instant the switch to the electrical supply is closed, the current will flow in the condenser circuit at a maximum rate. As the condenser becomes charged, there is an increase in the potential difference between the condenser surfaces; the rate of current flow decreases in the circuit, reaching zero as the condenser becomes fully charged; and the potential difference between the surfaces becomes equal and opposite to the voltage of the source. Figure 2–5 shows the circuit connections and variation in current while the condenser is being charged. The current will not reach zero if the dielectric between the surfaces is not a perfect insulator because a small leakage current will flow through the dielectric between the surfaces. If the dielectric is perfect, the condenser can be disconnected from the supply after it is charged and it will hold the stored electrical energy indefinitely. If, however, the dielectric permits leakage currents to flow, the charge will gradually leak away.

If an a-c supply is connected to the condenser terminals, a different current flow is observed. The conditions existing during the charging of a condenser can be expressed as

$$q = Ce$$

and

$$q = \int i\,dt$$

and, for a given condenser,

$$Ce = \int i\,dt$$

$$e = \frac{1}{C}\int i\,dt$$

$$i = C\frac{de}{dt}$$

If an alternating voltage of the value $e = E_m \sin \omega t$ is impressed on this condenser, the expression for the current which flows in the condenser becomes

$$i = C\frac{d}{dt}E_m \sin \omega t$$

Differentiating,

$$i = \omega C E_m \cos \omega t$$

or

$$i = \omega C E_m \sin\left(\omega t + \frac{\pi}{2}\right)$$

and

$$i = I_m \sin\left(\omega t + \frac{\pi}{2}\right)$$

where

$$I_m = \omega C E_m$$

It follows that the current in a condenser, when a sinusoidal voltage is impressed across its terminals, is a sinusoidal current which leads the impressed voltage by an angle of 90°.

As has been shown for resistance and inductance, there is an induced or opposition voltage which opposes the impressed voltage. The opposition voltage e_C opposes the impressed voltage e. If $e = \frac{1}{C}\int i\,dt$, then

$e_C = -\frac{1}{C}\int i\,dt$. The time relationships for the above are shown in Fig. 3–5.

If $I_m = \omega CE_m$, the opposition to the current flow or the impedance of the circuit must be equal to $1/\omega C$.

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi fC}$$

is called the capacitive reactance of the circuit and is expressed in ohms.

As for inductance, the opposition to current flow caused by a condenser is called reactance to distinguish it from the energy-consuming opposi-

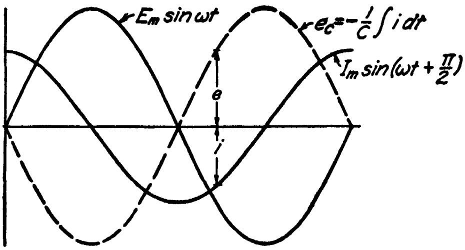


FIG. 3–5. Current and voltage waves for a circuit containing only a condenser. The current leads the voltage in time phase by 90 degrees.

tion, resistance. The current and voltage of a system containing only resistance are always in time phase with each other, but the current and voltage of a system containing only capacitive reactance are out of time phase with each other by 90 electrical degrees with the current leading the voltage.

If the expression

$$i = \omega CE_m \sin\left(\omega t + \frac{\pi}{2}\right)$$

is expanded to

$$i = 2\pi fCE_m \sin\left(2\pi ft + \frac{\pi}{2}\right)$$

it will be seen that the current of the circuit will vary directly as the frequency (f) when the value of E_m remains constant. In other words, the current of the circuit depends not only upon the voltage but also upon the frequency of the system. That is, the circuit opposition varies with the frequency. If the frequency is zero, the opposition to current flow is infinite.

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi(0)C} = \frac{1}{0} = \infty$$

and the current flow is zero. This is the condition of the current in a condenser when a direct current supply of voltage is impressed and the condenser is fully charged.

Example e. A condenser of 20 microfarads (μf) is connected to an a-c system. What is the reactance for this condenser if the system frequency is (*a*) 60 cycles? (*b*) 25 cycles?

$$X_{C_{60}} = \frac{1}{2\pi(60)20 \times 10^{-6}}$$

$$X_{C_{60}} = \frac{1 \times 10^6}{2\pi \times 60 \times 20} = \frac{10^6}{2400\pi} = \frac{10^4}{24\pi} = \frac{10^4}{75.4} = 132.5 \text{ ohms}$$

$$X_{C_{25}} = X_{C_{60}} \times \tfrac{60}{25} = 132.5 \times 2.4 = 318 \text{ ohms}$$

Example f. If a voltage, $e = 500 \sin 377t$, is impressed on a condenser of 20 μf, what will be the expression for the current of the circuit?

If $e = 500 \sin 377t$, the frequency equals

$$\frac{377}{2\pi} = 60 \text{ cycles}$$

$$X_C = 132.5 \text{ ohms (example } e)$$

$$i = \frac{500}{132.5} \sin\left(377t + \frac{\pi}{2}\right)$$

$$i = 3.77 \sin\left(377t + \frac{\pi}{2}\right)$$

5. Capacitances in Series and in Parallel. When several capacitances are connected in series, Fig. 4–5, the current is the same in each capaci-

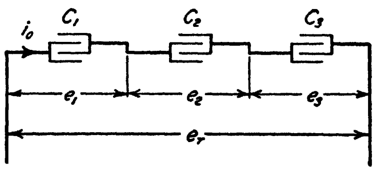


FIG. 4–5. A circuit consisting of capacitances connected in series.

tance. If this current is a sinusoidal current ($i = I_m \sin \omega t$) the voltage for each capacitance is

$$e_1 = \frac{1}{\omega C_1} I_m \sin\left(\omega t - \frac{\pi}{2}\right)$$

$$e_2 = \frac{1}{\omega C_2} I_m \sin\left(\omega t - \frac{\pi}{2}\right)$$

$$e_3 = \frac{1}{\omega C_3} I_m \sin\left(\omega t - \frac{\pi}{2}\right)$$

The total voltage is

$$e_T = e_1 + e_2 + e_3$$

Substituting,

$$e_T = \left(\frac{1}{\omega C_1} + \frac{1}{\omega C_2} + \frac{1}{\omega C_3}\right) I_m \sin\left(\omega t - \frac{\pi}{2}\right) = \frac{1}{\omega C_0} I_m \sin\left(\omega t - \frac{\pi}{2}\right)$$

or

$$\frac{1}{\omega C_0} = \frac{1}{\omega C_1} + \frac{1}{\omega C_2} + \frac{1}{\omega C_3}$$

or

$$X_{C_0} = X_{C_1} + X_{C_2} + X_{C_3}$$

also

$$\frac{1}{C_0} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

In a series circuit containing only capacitances, the total capacitive reactance is the numerical sum of the individual capacitive reactances of the circuit. The reciprocal of the total capacitance is equal to the sum of the reciprocals of the individual capacitances.

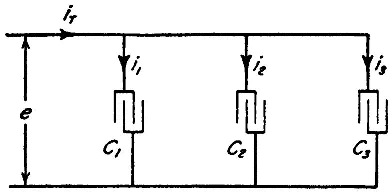


FIG. 5–5. A circuit consisting of capacitances connected in parallel.

When several capacitances are connected in parallel, Fig. 5–5, the voltage is the same for each capacitance. For a sinusoidal voltage ($e = E_m \sin \omega t$) across the capacitances, the currents in the capacitances are

$$i_1 = \omega C_1 E_m \sin\left(\omega t + \frac{\pi}{2}\right)$$

$$i_2 = \omega C_2 E_m \sin\left(\omega t + \frac{\pi}{2}\right)$$

$$i_3 = \omega C_3 E_m \sin\left(\omega t + \frac{\pi}{2}\right)$$

The total current

$$i_T = i_1 + i_2 + i_3$$

By substitution

$$i_T = (\omega C_1 + \omega C_2 + \omega C_3)E_m \sin\left(\omega t + \frac{\pi}{2}\right) = \omega C_0 E_m \sin\left(\omega t + \frac{\pi}{2}\right)$$

or

$$\omega C_0 = \omega C_1 + \omega C_2 + \omega C_3$$

and

$$C_0 = C_1 + C_2 + C_3$$

also

$$\frac{1}{X_{C_0}} = \frac{1}{X_{C_1}} + \frac{1}{X_{C_2}} + \frac{1}{X_{C_3}}$$

In a parallel circuit containing only capacitances, the reciprocal of the total capacitive reactance is equal to the sum of the reciprocals of the individual capacitive reactances of the circuit. The total capacitance is equal to the numerical sum of the individual capacitances of the circuit.

6. Energy Stored in a Condenser. During the time that a condenser is being charged, energy is required to raise the potential between the plates of the condenser. This energy is stored in the electrostatic field of the condenser and is released by the condenser at the time of discharge. Consequently, the average energy consumed by a condenser during a charge-discharge interval is zero. If a condenser having a charge q has its potential difference e increased by a small increment, the work done is

$$dw = q\,de = Ce\,de$$

The total work done in charging a condenser from an initial voltage, having a value $e = E_1$ at the instant of change in charging conditions to a final voltage $e = E_2$ for the charging conditions, can be expressed as

$$J = \int q\,de = \int_{E_1}^{E_2} Ce\,de$$

$$J = \frac{C(E_2^2 - E_1^2)}{2} \text{ joules}$$

or, where the initial potential or final potential on the condenser is zero,

$$J = \frac{CE^2}{2} \text{ joules}$$

If an alternating voltage is impressed on a condenser, the energy stored in the condenser for one-half cycle is zero, since the net change in voltage is zero. However, if the electric circuit is opened at any instant, the amount of energy stored in the condenser at that instant is available for

use in any manner desired. This principle of charging and discharging a condenser is a method used extensively for accurate timing whenever short intervals of time are to be recorded.

Example g. A condenser is charged by a d-c voltage of 500 volts. If the condenser has a capacity of 20 μf, how much energy is stored in the condenser when it is charged?

$$J = \tfrac{1}{2}CE^2$$

$$J = \frac{20 \times 500^2}{2 \times 10^6} = \frac{10 \times 250{,}000}{10^6} = \frac{25 \times 10^5}{10^6} = 2.5 \text{ joules}$$

PROBLEMS

1–5. A current, $i = 20 \sin 100\pi t$, flows in a condenser circuit. Determine the maximum charge which the condenser can have from this current.

2–5. A current, $i = 10 \sin 377t$, flows in a condenser circuit. When $t = 1/240$ sec, what is the charge on the condenser?

3–5. When a 20-microfarad condenser is connected to a 60-cycle system, a current, $i = 2.5 \sin (377t + 30°)$, flows. What is the sinusoidal expression for the voltage across the condenser? What would the voltmeter read?

4–5. A condenser having a capacity of 150 microfarads is fully charged from a 250-volt d-c supply. The condenser is disconnected from this source and then discharged through a resistance. If the time to discharge the condenser completely is 0.02 sec what is the average value of this current?

5–5. A voltage, $v = 150 \sin (377t + 45°)$, is impressed upon a 50-microfarad condenser. What is the sinusoidal expression for the current? What would an ammeter read in this circuit?

6–5. A 25-microfarad condenser and a 50-microfarad condenser are connected in parallel across a voltage $v = 100 \sin 377t$. What is the sinusoidal expression for the total current flowing?

7–5. If the condensers of Prob. 6 are connected in series across the same voltage, what would be the reading of an ammeter connected in the circuit?

8–5. If a condenser stores 10 joules of energy when connected to a 250-volt d-c source, what is the capacity of the condenser?

9–5. A current of 0.5 amp flows in a 25-microfarad condenser when connected to a 60-cycle source of voltage. What is the characteristic sinusoidal expression for a 25-cycle voltage required to give the same effective current? The current has the sinusoidal form $i = I_m \sin \omega t$.

10–5. A condenser of what capacity will give the same magnitude of reactance at 25 cycles as an inductance of 0.2 henry?

11–5. Determine the change in charge on a condenser during a half cycle if the current flow is sinusoidal and has the value $i = 100 \sin 377t$.

12–5. A voltage, $v = 200 \sin [100\pi t - (\pi/6)]$, is applied to a condenser of 79.5 microfarads. (*a*) What is the expression for the current in the circuit? (*b*) What is the maximum energy stored in the condenser?

13–5. A voltage, $v = 100 \sin [377t + (\pi/3)]$, is applied to a condenser of 20 microfarads. (*a*) What is the value of the current in the condenser when $t = 0.1$ sec? (*b*) What is the value of the instantaneous power at this instant?

14–5. A current, $i = 10 \sin 100\pi t$, is flowing in a condenser. The capacitive reactance of the condenser at this frequency is $10/\pi$ ohms. Determine the quantity of electricity stored in the condenser at the instant $t = 1/150$ sec.

15–5. When a current of 10^{-3} amp flows into a condenser for 1 sec the voltage across the condenser increases 100 volts. Determine the capacity of the condenser. What is the current (effective value) that flows when a 1000-cycle voltage having a maximum value of 10 volts is impressed on this condenser?

CHAPTER 6

THE GENERAL CIRCUIT

In this chapter the relationship is developed between the sine waves and their vectors and also between the voltage and current waves. The chapter also deals with the uses of the relationships between the opposition parameters (R, $X_{\mathfrak{L}}$, X_C). These parameters are considered as pure resistance, inductance, and capacitance, and the circuits are arranged with these pure parameters either in series or parallel. In circuits of pure parameters, the power factor and phase relationships appear in their significance.

In the previous chapters, in which fundamental oppositions have been considered, the symbols E and e have been used to designate every voltage, whether impressed or generated. To differentiate between generated (induced voltages, which cannot be measured with meters) and impressed (terminal voltages, which can be measured with meters) voltages, the first will be marked E or e and the latter V or v. Though the A.I.E.E. recommends the use of E with subscripts, for instruction purposes different symbols are preferable.

1. Statistical or Experimental Information. The foregoing chapters (Chapters 3, 4, 5) treated in detail the opposition factors in the electrical system. These factors act in the same manner whether they are parts of circuits or of machines. It has been shown that each opposition sets up a countervoltage and for this countervoltage, in order to overcome the opposition to current flow, a certain portion of the potential applied to the system is required. The magnitudes of the impressed voltages required by the various counteractions are

$$\text{Resistance} \qquad v_R = iR$$

$$\text{Inductance} \qquad v_{\mathfrak{L}} = \mathfrak{L}\frac{di}{dt}$$

$$\text{Capacitance} \qquad v_C = \frac{1}{C}\int i\,dt = \frac{q}{C}$$

The countervoltages are all negative with respect to the applied voltage. However, no voltage can be negative except in a relative sense

and each of the countervoltages may be a source of potential with respect to some new electrical system or portion of the initial system. The positive and negative signs, as applied to source and sink in the study of electrical phenomena, depend upon viewpoint and reference and, when the reference is once established, care must be taken that it is not changed. A new problem may have a different base as a reference, and the reference may be changed at any time in a problem if all concepts are changed accordingly.

There are two laws applying to electrical systems which are similar to the two basic laws governing all natural phenomena. These laws in general are: (1) *energy can be neither created nor destroyed* and (2) *to every action there is an equal and opposite reaction.* These laws, transposed into mathematical expressions, make possible the solution of many problems in the various branches of engineering and are widely used. Often it is difficult to form the equations and still more difficult to account for the various parts, but a successful solution of engineering problems depends upon the comprehension of the conditions necessary to satisfy the above fundamental laws.

The laws of electrical networks, stated by Kirchhoff (1857), are

1. The algebraic sum of the instantaneous currents flowing toward a junction in a network is equal to zero.
2. The algebraic sum of the instantaneous voltage drops in any closed path of a network is equal to zero. Voltage rise is a negative voltage drop.

These laws, as stated by Kirchhoff, originally were applied to direct current but they may be applied (with equal accuracy) to instantaneous values of alternating current. The laws have been expanded to make them applicable to both systems.

2. Basic Analysis. Of equal importance in a complete understanding of the group of accepted (rather than developed) fundamentals are the definitions for the various terms. The foregoing paragraphs summarize the fundamental considerations in the electrical system that fall under the classification of statistical facts. If the general laws of nature are expressed in mathematical form, numerous relationships are brought out which can be established by experimental processes. These are the relationships used in the discussion of all machine characteristics and circuit solutions.

Figure 1–6 shows a simple series circuit in which all the opposition factors are represented. It may be assumed that either of two types of current will flow in the system after a switch has been closed and the system has reached a steady state. The current may be a direct current of constant value (as obtained from a battery) or a sinusoidal current

(as obtained from an a-c generator). The circuit will be studied to determine what wave of voltage must be applied to produce an assumed current flow. The solution will be based on the use of general fundamental relationships in the electrical circuit and mathematical developments of the simpler form.

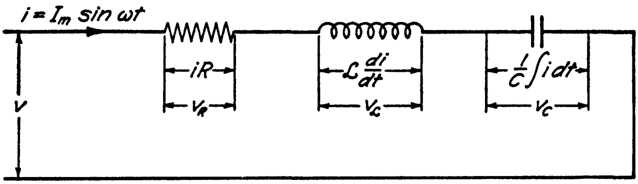


FIG. 1–6. A series circuit containing pure resistance, inductance, and capacitance.

3. Potential and Current Relationships. Applying Kirchhoff's Second Law to the electrical system of Fig. 1–6, all the applied voltage will be absorbed in the various counteractions in the system, and the algebraic expression for the law will be

$$v = v_R + v_{\mathcal{L}} + v_C$$

$$v = iR + \mathcal{L}\frac{di}{dt} + \frac{1}{C}\int i\,dt \ *$$

The two current conditions to be studied are

$$i = I_m$$

and

$$i = I_m \sin \omega t$$

A steady-state condition of $i = I_m$ cannot be established by a d-c voltage because the condenser introduces into the system an open circuit; and, if there is no leakage in the condenser, there will be no current flow after the initial charge. Therefore, the applied voltage produces a zero steady-state current.

If the current is sinusoidal, the applied voltage is much more difficult to calculate and new definitions are introduced into the electrical vocabulary. Applying a voltage to the system in Fig. 1–6 produces the following result.

$$v = iR + \mathcal{L}\frac{di}{dt} + \frac{1}{C}\int i\,dt$$

* A complete solution contains both steady-state and transient expressions; only steady-state expressions are considered here. The treatment assumes constant frequency, sinusoidal wave motion in the a-c supply, and absolutely constant conditions in the d-c supply. Constant permeability and steady-state operation are assumed in both cases.

If

$$i = I_m \sin \omega t$$

$$v = I_m R \sin \omega t + I_m \mathfrak{L} \frac{d}{dt} (\sin \omega t) + \frac{I_m}{C} \int \sin \omega t \, dt$$

$$v = I_m \left[\left(R \sin \omega t + \mathfrak{L}\omega \cos \omega t - \frac{1}{\omega C} \cos \omega t \right) \right]$$

This indicates that the applied voltage must be made up of the sum of three sine waves. It is possible to simplify the expression mathematically.

The last two terms may be combined into one by removing the common function of the angle so that

$$v = I_m \left[R \sin \omega t + \left(\mathfrak{L}\omega - \frac{1}{\omega C} \right) \cos \omega t \right] \qquad (a\text{–}6)$$

which reduces the number of sine waves to two, and the applied voltage is now expressed by the sum of these two sine waves.

As shown in Chapters 4 and 5, the value of

$$\omega \mathfrak{L} = 2\pi f \mathfrak{L} = X_{\mathfrak{L}}$$

and

$$\frac{1}{\omega C} = \frac{1}{2\pi f C} = X_C$$

where f is the frequency, $\mathfrak{L}$ is the inductance in henrys, and C is the capacitance in farads. The values of $\omega\mathfrak{L}$ and $1/\omega C$ are replaced by terms called the inductive and capacitive reactance. When the expression is in the form of a coefficient and an angular velocity, it is the common form used in the field of energy engineering (or communication); reactance values are used by the power engineer; however, to some extent both are used in each field.

The coefficient of the second sine wave then becomes

$$\omega \mathfrak{L} - \frac{1}{\omega C} = X_{\mathfrak{L}} - X_C$$

$$X_{\mathfrak{L}} - X_C = \pm X$$

where the sign designates the kind of resultant reactance. The term X, known as the equivalent reactance of the system, is measured in ohms. Since the capacitance and the inductance are opposite in their effect, to designate its character, the equivalent reactance may have either a negative or positive sign, depending upon which predominates (Fig. 2–6).

In power considerations, capacitance rarely predominates, the exceptions being in very lightly loaded transmission lines and the special performance of synchronous motors.

If the value X is substituted in the expression for v in equation (a–6), it takes the form

$$v = I_m(R \sin \omega t \pm X \cos \omega t)$$

which may be reduced to represent a single sine wave. To reduce these two waves to a single sine wave, it is necessary to introduce another angle and the functions of that angle. This angle may be either positive or negative; that is, turned in a counterclockwise or a clockwise direction.

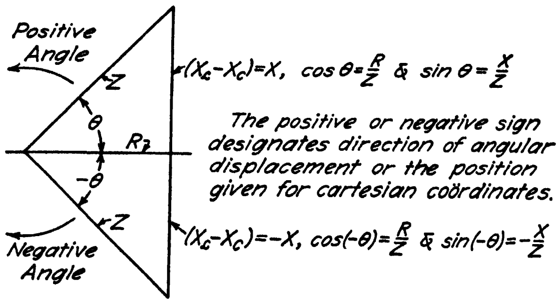


FIG. 2–6. Impedance triangles showing the relationship between the parameters of the circuit.

tion. If the inductance predominates, the angle will be turned counterclockwise; if the capacitance predominates, the angle will be turned clockwise (this applies to the impedance diagram). This same relationship holds if the voltage is given with reference to the current. Figure 2–6 shows the right triangles constructed under the two conditions, with the functions of the angles and the geometric relationships of the sides as follows.

$$\pm\,\theta = \text{angular displacement or } \textit{power factor angle}\ \dagger$$

$$\cos \pm\,\theta = \frac{R}{Z} \quad \textit{power factor}\ \dagger\ddagger$$

$$\sin \pm\,\theta = \pm\,\frac{X}{Z} \quad (\textit{reactive factor}\ \dagger\ddagger)$$

$$Z = \sqrt{R^2 + (\pm X)^2} \quad \textit{impedance}\ \dagger$$

† The positive or negative sign designates direction of angular displacement or the position given for cartesian coordinates.

‡ In Chapter 11 the power factor and reactive factor are developed as operators upon active and reactive power respectively.

The characteristics of the various terms are the best definitions that could be given for them. The power factor will be shown to bear a definite relationship to the actual power demanded by the electrical system. The fact that the resistance and the reactance are in effect at 90° to each other has been shown as a mathematical necessity but, in the laboratory, it can be proved in a physical sense.

If the terms in the expression of voltage v are multiplied by Z/Z, it takes the form

$$v = I_m Z \left(\frac{R}{Z} \sin \omega t \pm \frac{X}{Z} \cos \omega t \right)$$

and it will at once be evident that R/Z and X/Z are functions of the angle θ shown in Fig. 2–6. By substituting these functions, the form is

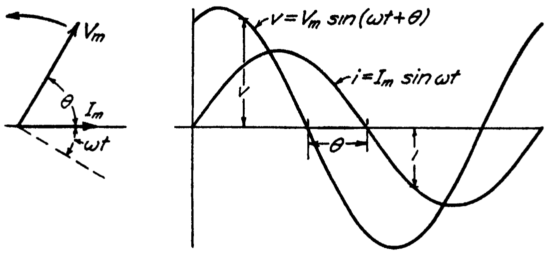


FIG. 3–6. Sinusoidal voltage and current waves in a series system containing pure resistance, inductance, and capacitance, inductance predominating.

that of the sum and difference of two angles which permits the replacement of the two sine waves by a single sine wave, or

$$v = I_m Z(\cos \theta \sin \omega t \pm \sin \theta \cos \omega t)$$

$$v = I_m Z \sin (\omega t \pm \theta)$$

which is another sinusoidal wave displaced from the current wave by an angle θ, with the voltage wave either ahead or behind the current wave (leading or lagging), depending upon whether the sign is plus or minus. The nature of the sign depends upon the kind of reactance. A predominance of inductance causes the impressed voltage to lead the current, whereas the reverse is true when the capacitive reactance predominates.

The instantaneous voltage becomes a maximum when the sin $(\omega t \pm \theta)$ is unity. Then in algebraic form

$$V_m = I_m Z$$

which is often called the Ohm's Law of the a-c circuit. If each side of the expression is divided by $\sqrt{2}$, then,

$$V = IZ \quad \text{in effective values}$$

Substituting the value for V_m in the instantaneous value for voltage the final form is

$$v = V_m \sin(\omega t \pm \theta)$$

which is a wave with a generating vector, V_m in length, and a uniform angular velocity ω. Figure 3–6 shows the relationship of the sinusoidal waves of potential and current.

4. Analysis of the Resultant Sinusoidal Waves of Voltage and Current in a Series System. It has been found that the expressions for current and voltage in a series system are

$$i = I_m \sin \omega t \tag{b–6}$$

$$v = V_m \sin(\omega t \pm \theta) \tag{c–6}$$

or, that it takes the voltage given in equation (c–6) to cause the sinusoidal current in equation (b–6) to flow.

In a system where consideration is given to resistance R, inductive reactance X_L, and capacitive reactance X_C, there are seven possible combinations:

1. Resistance only.
2. Inductive reactance only.
3. Capacitive reactance only.
4. Resistance and inductive reactance.
5. Resistance and capacitive reactance.
6. Inductive and capacitive reactance.
7. Resistance, inductive, and capacitive reactance.

Combinations 4 and 5 fulfill the conditions of 7, since capacitive and inductive reactance are opposite in sign and only the resultant effect need be considered.

Table I–6 gives, in a condensed form, the resultant effects for these seven parameter combinations, whereas Figs. 4–6 to 7–6 show the resultant conditions, in graphical form.

5. A System Containing Resistance. When the system has only resistance (Chapter 3), the power factor will be the cosine of a zero angle, because the ratio of the resistance to the impedance (which is now equal to the resistance) is 1. The instantaneous values of current and voltage are

$$i = I_m \sin \omega t$$

$$v = V_m \sin \omega t$$

which are two sine waves in phase with each other as shown by Fig. 4–6. Frequently only resistance is considered in the alternating circuit for,

TABLE I-6

RELATIONSHIPS BETWEEN FACTORS INVOLVED IN THE SERIES CIRCUIT

Circuit Types (p. 104)	1	2	3	4	5	6	7
Parameters	R	$\mathcal{L}$	C	R and $\mathcal{L}$	R and C	$\mathcal{L}$ and C [1]	R, $\mathcal{L}$, and C [1]
R	R	0	0	R	R	0	R
$X_{\mathcal{L}}$	0	$2\pi f\mathcal{L}$	0	$2\pi f\mathcal{L}$	0	$2\pi f\mathcal{L}$	$2\pi f\mathcal{L}$
X_C	0	0	$\frac{1}{2\pi fC}$	0	$\frac{1}{2\pi fC}$	$\frac{1}{2\pi fC}$	$\frac{1}{2\pi fC}$
θ	0	90°	−90°	>0, <90°	<0, >−90°	90° or −90°	<90°, >−90° but not 0
$\cos\theta$ Power Factor	1	0	0	>0, <1 but not 0	>0, <1 but not 0	0	<1 but not 0
v	$V_m \sin \omega t$	$V_m \sin\left(\omega t + \frac{\pi}{2}\right)$	$V_m \sin\left(\omega t - \frac{\pi}{2}\right)$	$V_m \sin(\omega t + \theta)$	$V_m \sin(\omega t - \theta)$	$V_m \sin\left(\omega t \pm \frac{\pi}{2}\right)$	$V_m \sin(\omega t \pm \theta)$
X	0	$X_{\mathcal{L}}$	X_C	$X_{\mathcal{L}}$	X_C	$X_{\mathcal{L}} - X_C$	$X_{\mathcal{L}} - X_C$
Z	R	$X_{\mathcal{L}}$	X_C	$\sqrt{R^2 + X_{\mathcal{L}}^2}$	$\sqrt{R^2 + X_C^2}$	$X_{\mathcal{L}} - X_C$	$\sqrt{R^2 + (X_{\mathcal{L}} - X_C)^2}$
Phase	V and I in phase	V leads I by 90°	V lags I by 90°	V leads I by θ	V lags I by θ	V leads or lags I by 90°	V leads or lags I by θ

[1] Series (phase) resonance considered in Chapter 8.

when the reactance is low and may be neglected, the circuit is one of pure resistance. In the analysis of the system (even when it contains an appreciable reactance), it is not uncommon to consider the system by parts, the resistance component of voltage being placed in phase with the current and the reactance components in quadrature with the current.

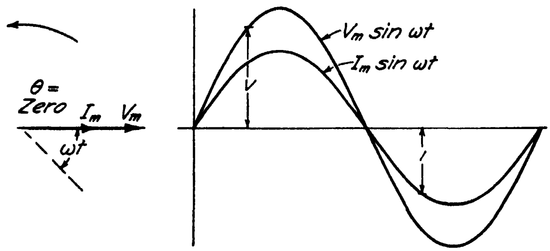


FIG. 4–6. Sinusoidal waves of voltage and current in a system containing pure resistance.

6. A System Containing Only Reactance. When the system contains only reactance (Chapters 4 and 5), the voltage is in quadrature (90°) with the current. Figures 5–6 and 6–6 show the resultant effect of inductive and capacitive reactance, respectively. The angle being 90°, the cosine of the power factor angle is zero in both instances. In a system containing both inductive and capacitive reactance, the current will either lag or lead the voltage by 90°, depending upon whether the inductive or the capacitive reactance predominates. Where the inductive

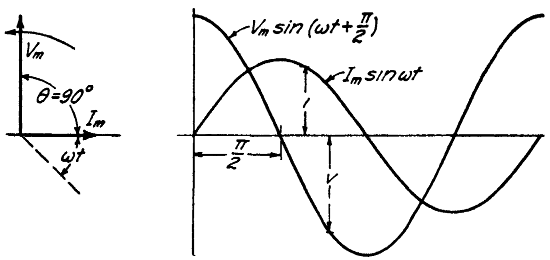


FIG. 5–6. Sinusoidal waves of voltage and current in a system containing pure inductance.

and the capacitive reactance are equal *resonance* exists (Chapter 8). Neither of these conditions is common in the treatment of electrical equipment or secondary circuits§ but, as in resistance, it is frequently desirable to consider the parts separately and study the reactance of

§ A study of transmission systems is often based on the reactance values of the system.

each. Though pure resistance and pure reactance are strictly hypothetical, they are of prime analytical importance.

7. A System Containing Both Resistance and Reactance. The most common of the resistance-reactance type of circuit is the one containing inductive reactance. In most machines and in all secondary circuits,

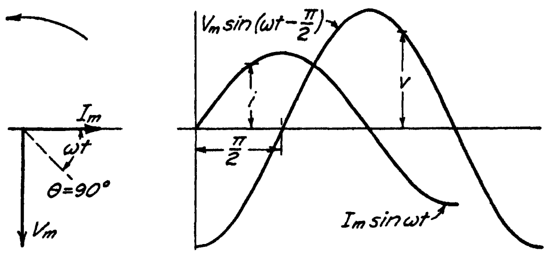


FIG. 6–6. Sinusoidal waves of voltage and current in a system containing pure capacitance.

the capacitance is very small and therefore of minor importance. In long cables and lightly loaded transmission lines, the capacitance may predominate and be the controlling factor in the characteristics and operation.

Figures 3–6 and 7–6 show the system being controlled by the inductance and capacitance, respectively. One of the conditions shown will be suitable for any circuit containing all three types of counteraction, since either the inductance or the capacitance will predominate. In the

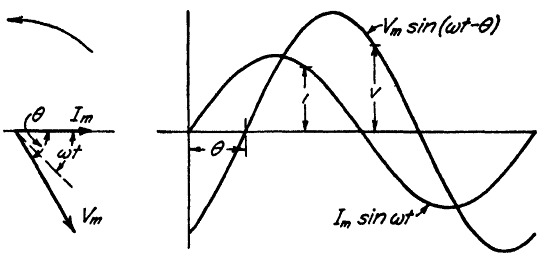


FIG. 7–6. Sinusoidal waves of voltage and current in a system containing resistance, inductance, and capacitance in series, capacitance predominating.

event the system contains equal amounts of inductive and capacitive reactance, there exists a phenomenon known as resonance, when the system operates as if it were a pure resistance.

Example a. On a 60-cycle system, a pure resistance of 2 ohms and a pure inductive reactance and a pure capacitive reactance of 2 and 1 ohms, respectively, are placed in series across a sinusoidal voltage which will cause a maximum

current of 10 amp to flow. What will be the instantaneous value of voltage 0.01 sec after the current wave has passed through zero approaching a positive maximum?

$$v = I_m Z \sin(\omega t + \theta) \quad \text{(inductance predominates)}$$

$$Z = \sqrt{2^2 + (2-1)^2} = 2.24 \text{ ohms}$$

$$\cos\theta = \frac{2}{2.24} = 0.893$$

$$\theta = 26.8°$$

$$v = 10 \times 2.24 \sin(377 \times 0.01 + \theta)$$

$$= 22.4 \sin 242.8°$$

$$v = -19.9 \text{ volts}$$

8. The D-C System. It was indicated in Art. 3 that a d-c system containing a pure condenser is an open circuit, or a circuit which contains an infinite opposition. Whenever the condenser is not present and only

TABLE II–6

FACTORS INVOLVED IN THE D-C SERIES CIRCUIT

Circuit Type (p. 104)	1	2	3	4	5	6	7
Parameters	R	$\mathfrak{L}$	C	R and $\mathfrak{L}$	R and C	$\mathfrak{L}$ and C	R, $\mathfrak{L}$, and C
R	R	0	0	R	R	0	R
$X_{\mathfrak{L}}$	0	0	0	0	0	0	0
X_C	0	0	Infinite	0	Infinite	Infinite	Infinite
i	I_m	Infinite	0	I_m	0	0	0
v	iR	Indeterminate	Indeterminate	iR	Indeterminate	Indeterminate	Indeterminate

inductance and resistance are present, the system conditions may be tabulated in a fashion similar to the table given for the a-c system. Table II–6 shows the existing relationships in the presence of counter-

actions. A close analysis of the table shows that the only factor that need be considered in the d-c system is resistance, because the voltage reduces to

$$v = iR$$

with i as a constant. This is known as Ohm's Law (1827) and is directly applicable to systems supplied by batteries and d-c generators. In rectifiers, the wave ripple must be reduced to a very small value before this law can be safely applied, since the impulse delivered by most rectifiers is influenced by inductance and capacitance between leads. If the impulse wave is smoothed by means of suitable filters, Ohm's Law is applicable.

Example b. To a d-c circuit are added, in series, in the following order, a pure resistance, a pure inductance, and a pure capacitance of 2 ohms, 2 henrys, and 1 microfarad, respectively. Voltage is applied to give a maximum current of 10 amp. What will be the voltage across each part of the circuit as each opposition is added?

A pure resistance placed across the voltage:

$$v = iR$$

$$i = I_m = 10$$

$$v = 10 \times 2 = 20 \text{ volts}$$

A pure inductance is added to the system which now contains a resistance and inductance in series:

$$v = \mathcal{L}\frac{di}{dt}$$

$$i = I_m = 10$$

$$v = \mathcal{L}\frac{d}{dt}(I_m)$$

$$v = 0, \text{ across the inductance}$$

$$v = 20 + 0 = 20 \text{ volts}$$

A pure capacitance is added to the system which now contains a resistance, an inductance, and a capacitance in series:

$$v = \frac{1}{C}\int i\,dt$$

$$i = I_m = 0$$

$$v \text{ is indeterminate}$$

which, from a physical viewpoint, means that after the initial charging current of the system (during the transient period of the operating cycle) no current can flow, regardless of the voltage.

9. Analysis of the Parallel Electrical System using Kirchhoff's First Law. In the study of the series system, it was necessary to express statistical relationships in the form of voltages to permit the use of Kirchhoff's Second Law. The magnitude of the current which will flow through the various counteractions when they are in parallel is

$$\text{Resistance} \qquad i_R = \frac{v}{R}$$

$$\text{Inductance} \qquad i_{\mathcal{L}} = \frac{1}{\mathcal{L}}\int v\,dt$$

$$\text{Capacitance} \qquad i_C = C\frac{dv}{dt}$$

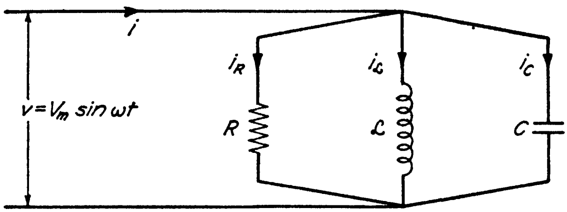


FIG. 8–6. A circuit containing pure resistance, inductance, and capacitance in parallel.

Figure 8–6 shows the connection of the different counteractions in parallel and the currents which will flow in each. The summation of the currents at the junction will give the following expression.

$$i = i_R + i_{\mathcal{L}} + i_C$$

$$i = \frac{v}{R} + \frac{1}{\mathcal{L}}\int v\,dt + C\frac{dv}{dt} \qquad (d\text{–}6)$$

which states that the instantaneous current entering any junction is equal to the current leaving that junction at the same instant. Both d-c voltage and a-c voltage may be applied to the parallel system and are expressed respectively by

$$v = V_m$$

and

$$v = V_m \sin \omega t$$

Applying the first condition, the d-c voltage, the last two terms of the equation (d–6) will be a practical impossibility. That is, the condenser now acting as an open circuit will not permit a flow of current and the inductance acts as a short circuit, because a zero resistance offers no

opposition to the flow of current. A d-c voltage applied to such a system will permit an infinite flow of current and cause destruction to the system unless it is protected. The general system, like the series circuit, will not function on a d-c system. In practice, the inductance branch has resistance and it is only this resistance that limits the current. If resistance only is placed across the system, the expression takes the form

$$i = \frac{v}{R}$$

which is another way of expressing Ohm's Law.

10. The A-C System with Parallel Branches. A sinusoidal voltage may be applied to the parallel system:

$$v = V_m \sin \omega t$$

which, when substituted in equation (d–6), gives

$$i = \frac{V_m}{R} \sin \omega t + \frac{1}{\mathfrak{L}} \int V_m \sin \omega t \, dt + C \frac{d}{dt} (V_m \sin \omega t)$$

Simplifying and collecting the above terms as in the series circuit, the expression reduces to

$$= V_m \left[\frac{1}{R} \sin \omega t + \left(\omega C - \frac{1}{\omega \mathfrak{L}} \right) \cos \omega t \right]$$

In this expression, as in the expression for the series system, the introduction of the following combinations has given rise to special definitions:

Conductance $\quad g = \dfrac{R}{Z^2} = \dfrac{R}{[\sqrt{R^2 + (X_{\mathfrak{L}} - X_C)^2}]^2}$

Inductive susceptance $\quad b_{\mathfrak{L}} = \dfrac{X_{\mathfrak{L}}}{Z^2} = \dfrac{X_{\mathfrak{L}}}{[\sqrt{R^2 + (X_{\mathfrak{L}} - X_C)^2}]^2}$

Capacitive susceptance $\quad b_C = \dfrac{X_C}{Z^2} = \dfrac{X_C}{[\sqrt{R^2 + (X_{\mathfrak{L}} - X_C)^2}]^2}$

These are the general terms for conductance and susceptance; they do not represent the values for the pure parameters given in Fig. 8–6.

It is necessary to examine the physical significance of circuit parameters to appreciate the true meaning of conductance and susceptance. These are not physical properties, but the result of mathematical analysis and convenience. Zero inductance ($\mathfrak{L}$) leads to zero inductive reactance ($X_{\mathfrak{L}}$), but zero capacitance (C) leads to infinite capacitive reactance (X_C). The capacitance must be infinite if the capacitive reactance is to be zero.

If a circuit with a condenser (two conductors separated by a perfect dielectric) has zero capacitance, no current can flow, for the system is open-circuited and the reactance is infinite.

To determine the values for the three branches shown in Fig. 8–6, the actual parameters must be substituted in the general forms given above, with R, $X_\mathcal{L}$, and X_C evaluated, where R, $\mathcal{L}$, and C are not shown in each branch, but where each branch contains a pure parameter.

BRANCH	CONDUCTANCE	SUSCEPTANCE
R	$g = \dfrac{R}{[\sqrt{R^2 + (0 - 0)^2}]^2} = \dfrac{1}{R}$	$b_\mathcal{L} = \dfrac{0}{[\sqrt{R^2 + (0 - 0)^2}]^2} = 0$
		$b_C = \dfrac{0}{[\sqrt{R^2 + (0 - 0)^2}]^2} = 0$
$\mathcal{L}$	$g = \dfrac{0}{[\sqrt{0^2 + (X_\mathcal{L} - 0)^2}]^2} = 0$	$b_\mathcal{L} = \dfrac{X_\mathcal{L}}{[\sqrt{0^2 + (X_\mathcal{L} - 0)^2}]^2} = \dfrac{1}{X_\mathcal{L}} = \dfrac{1}{\omega\mathcal{L}}$
		$b_C = \dfrac{0}{[\sqrt{0^2 + (X_\mathcal{L} - 0)^2}]^2} = 0$
C	$g = \dfrac{0}{[\sqrt{0^2 + (0 - X_C)^2}]^2} = 0$	$b_\mathcal{L} = \dfrac{0}{[\sqrt{0^2 + (0 - X_C)^2}]^2} = 0$
		$b_C = \dfrac{X_C}{[\sqrt{0^2 + (0 - X_C)^2}]^2} = \dfrac{1}{X_C} = \omega C$

The general terms for the conductance and susceptance are included in the foregoing definitions to eliminate the confusion associated in dealing with the specific example of pure parameters as compared with the general parameters. It must be kept clearly in mind that resultant expressions are applied to pure resistance, inductance, and capacitance branches. The resultant terms for a combination of the factors are treated in a succeeding chapter. Substituting the symbols for susceptance and conductance, the equation for current takes the form

$$i = V_m(g \sin \omega t \pm b \cos \omega t) \tag{e–6}$$

where the total susceptance of the system is a combination of the inductive and capacitive susceptance:

$$(b_C - b_\mathcal{L}) = \pm b$$

and the sign designates the type of the resultant susceptance (Fig. 9–6). Again, similar to series circuits, the resultant current wave has been reduced to two sine waves, and the next step is to contract these two waves into a single wave.

The procedure is the same as in the series system development. It will be necessary to introduce the functions of another angle, the angle of a right triangle formed by using g as a base and b as the altitude. Figure 9–6 shows the relationship between the various parts and the functions of the angle which has been introduced.

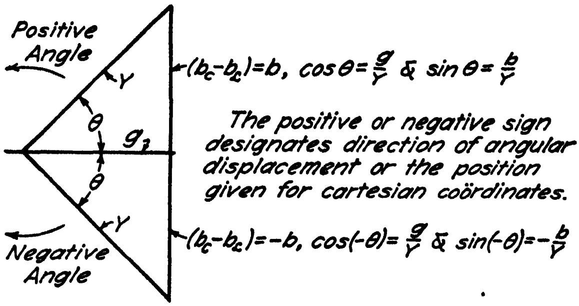


FIG. 9–6. Admittance triangles, showing the relationships between the parameters of the circuit.

$$\pm\theta = \text{angular displacement or } \textit{power factor angle} \,\|$$

$$\cos \pm \theta = \frac{g}{Y} \quad \textit{power factor} \,\| \,\P$$

$$\sin \pm \theta = \pm \frac{b}{Y} \quad \textit{reactive factor} \,\| \,\P$$

$$Y = \sqrt{g^2 + (\pm b)^2} \quad \textit{admittance} \,\|$$

If the equation (e–6) is multiplied by Y/Y, the instantaneous current expression becomes

$$i = V_m Y \left(\frac{g}{Y} \sin \omega t \pm \frac{b}{Y} \cos \omega t\right)$$

and, if the functions of the angle are substituted,

$$i = V_m Y(\cos\theta \sin \omega t \pm \sin\theta \cos \omega t)$$

When the sum or difference of the angle in the brackets is substituted, the expression becomes

$$i = V_m Y[\sin (\omega t \pm \theta)] \qquad (f\text{–}6)$$

|| The positive or negative sign designates direction of angular displacement or the position given for Cartesian coordinates.

¶ In Chapter 11 the power factor and reactive factor are developed as operators upon active and reactive power respectively.

In the development, the term Y (to be called admittance) has been introduced and it will be shown later that

$$Y = \frac{1}{Z} = \sqrt{g^2 + (b_C - b_L)^2}$$

The angle θ (shown in Fig. 9–6) is the power factor angle; and $\cos\theta$ is the power factor regardless of whether the value is obtained from conductance and admittance or from resistance and impedance.

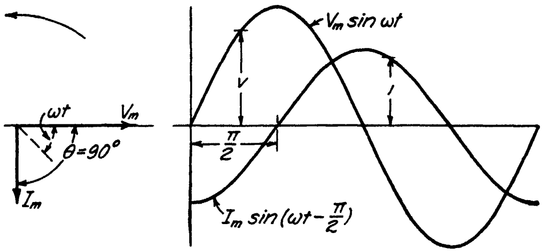


FIG. 10–6. Sinusoidal waves of voltage and current in a system containing pure inductance.

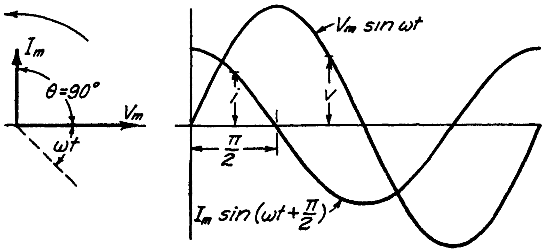


FIG. 11–6. Sinusoidal waves of voltage and current in a system containing pure capacitance.

The instantaneous current will be a maximum when $\sin(\omega t \pm \theta)$ is unity. Therefore,

$$I_m = V_m Y$$

When this is substituted in equation (f–6), the resultant expression is

$$i = I_m \sin(\omega t \pm \theta)$$

or, the wave form of the current will be sinusoidal and will either lead or lag the voltage wave by an angle θ, depending on whether capacitance or inductance is the controlling factor in the circuit or electrical system. Table III–6 gives the various combinations possible in the parallel system and the resultant current wave if sinusoidal voltage is impressed upon the system. Table IV–6 shows the combinations for the d-c system. Figures 4–6, 10–6, and 11–6 represent either parallel or series

TABLE III–6

RELATIONSHIPS BETWEEN FACTORS INVOLVED BY PURE PARAMETERS IN PARALLEL ON ALTERNATING CURRENT

Circuit Types (p. 104)	1	2	3	4	5	6	7
Parameters	R	$\mathcal{L}$	C	R and $\mathcal{L}$	R and C	$\mathcal{L}$ and C [1]	R, $\mathcal{L}$, and C [1]
g	$R/Z^2 = 1/R$	0	0	R/Z^2	R/Z^2	0	R/Z^2
$b_{\mathcal{L}}$	0	$\frac{X_{\mathcal{L}}}{Z^2} = \frac{1}{X_{\mathcal{L}}}$	0	$\frac{X_{\mathcal{L}}}{Z^2}$	0	$\frac{X_{\mathcal{L}}}{Z^2}$	$\frac{X_{\mathcal{L}}}{Z^2}$
b_C	0	0	$\frac{X_C}{Z^2} = \frac{1}{X_C}$	0	$\frac{X_C}{Z^2}$	$\frac{X_C}{Z^2}$	$\frac{X_C}{Z^2}$
θ	0	$-90°$	$90°$	$<0, >-90°$	$>0, <90°$	$-90°$ or $90°$	$>90°, <-90°$ not 0
$\cos\theta$	1	0	0	$>0, <1$ but not 0	$>0, <1$ but not 0	0	<1 but not 0
				$v = V_m \sin \omega t$			
i	$I_m \sin \omega t$	$I_m \sin \left(\omega t - \frac{\pi}{2}\right)$	$I_m \sin \left(\omega t + \frac{\pi}{2}\right)$	$I_m \sin (\omega t - \theta)$	$I_m \sin (\omega t + \theta)$	$I_m \sin \left(\omega t \pm \frac{\pi}{2}\right)$	$I_m \sin (\omega t \pm \theta)$
Y	g	$b_{\mathcal{L}}$	b_C	$\sqrt{g^2 + b_{\mathcal{L}}^2}$	$\sqrt{g^2 + b_C^2}$	$(b_C - b_{\mathcal{L}})$	$\sqrt{g^2 + (b_C - b_{\mathcal{L}})^2}$
b	0	$b_{\mathcal{L}}$	b_C	$b_{\mathcal{L}}$	b_C	$(b_C - b_{\mathcal{L}})$	$(b_C - b_{\mathcal{L}})$
Phase	I and V in phase	I lags V by 90°	I leads V by 90°	I lags V by θ	I leads V by θ	I lags or leads V by 90°	I lags or leads V by θ

[1] Parallel (phase) resonance considered in Chapter 9.

TABLE IV–6

RELATIONSHIPS BETWEEN FACTORS INVOLVED BY PURE PARAMETERS IN PARALLEL ON DIRECT CURRENT

Circuit Types (p. 104)	1	2	3	4	5	6	7
Parameters	R	$\mathcal{L}$	C	R and $\mathcal{L}$	R and C	$\mathcal{L}$ and C	R, $\mathcal{L}$, and C
g	$1/R$	0	0	0	$1/R$	0	$1/R$
$b_{\mathcal{L}}$	0	Infinite	0	Infinite	0	Infinite	Infinite
b_C	0	0	0	0	0	0	0
y	$1/R$	Infinite	0	Infinite	$1/R$	Infinite	$1/R$
i	I_m	Infinite	0	Infinite	I_m	Infinite	Infinite
e	iR	Indeterminate	Indeterminate	Indeterminate	iR	Indeterminate	Indeterminate

systems containing pure resistance, inductance, and capacitance, respectively. The only difference is in the statement of the phase relationship for, in the series circuit, the current is taken as the reference

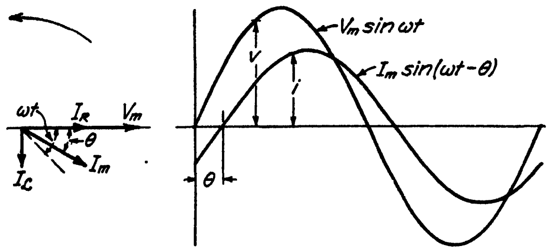


FIG. 12–6. Sinusoidal waves of voltage and current in a system containing pure resistance and inductance in parallel.

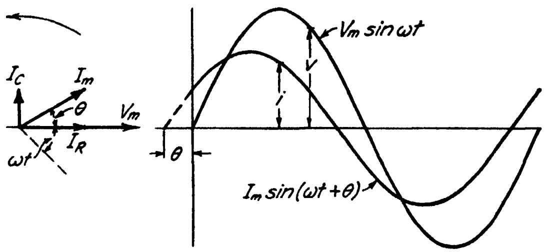


FIG. 13–6. Sinusoidal waves of voltage and current in a system containing pure resistance and capacitance in parallel.

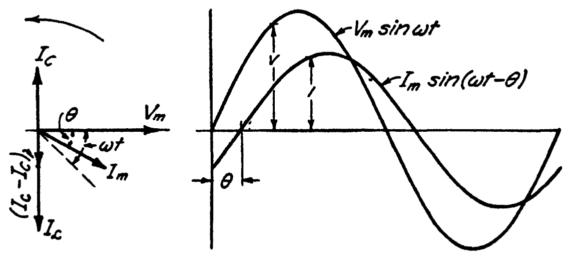


FIG. 14–6. Sinusoidal waves of voltage and current in a system containing resistance, inductance, and capacitance in parallel, inductive current predominating.

whereas, in the parallel circuit, the voltage is taken as the reference. Figures 12–6 and 13–6 show the combination of sine waves that gives the resultant current wave; Fig. 14–6 shows the general case, where the three sine waves of current in the various branches give the resultant current furnished by the generating system.

Example c. On a 60-cycle system, a 2-ohm resistance, a 2-ohm pure inductive reactance, and a 1-ohm pure capacitive reactance are placed in parallel across a sinusoidal voltage with a maximum voltage of 100 volts. What will the instantaneous current be at the end of 0.01 sec if, at zero time, the voltage passes through zero approaching a positive maximum?

$$i = V_m Y \sin(\omega t + \theta)$$

The positive sign is used with θ because the capacitive reactance predominates. More current will flow through the lesser opposition.

Resistance branch:

$$g = \frac{R}{Z^2} = \frac{1}{R} = \frac{1}{2} = 0.5 \qquad b_{\mathfrak{L}} = 0 \qquad b_C = 0$$

Inductance branch:

$$b_{\mathfrak{L}} = \frac{X_{\mathfrak{L}}}{Z^2} = \frac{1}{X_{\mathfrak{L}}} = \frac{1}{2} = 0.5 \qquad g = 0 \qquad b_C = 0 \quad \text{(sign of } b_{\mathfrak{L}} \text{ negative)}$$

Capacitance branch:

$$b_C = \frac{X_C}{Z^2} = \frac{1}{X_C} = \frac{1}{1} = 1 \qquad g = 0 \qquad b_{\mathfrak{L}} = 0$$

The capacitive and inductive susceptance are both positive in value but, in the solution for the admittance, they are relatively negative with respect to each other. Previous analysis has considered the inductive susceptance negative.

$$Y = \sqrt{0.5^2 + (1 - 0.5)^2} = 0.707$$

$$\cos\theta = \frac{0.5}{0.707} = 0.707$$

$$\theta = 45°$$

$$i = 100 \times 0.707 \times \sin(3.77 + \theta)$$

$$i = 70.7 \sin 261°$$

$$i = 70.7 \times (-0.988) = -69.9 \text{ amp}$$

Example d. Across a d-c circuit are placed, consecutively, a pure resistance of 2 ohms, a pure inductance of 2 henrys, and a capacity of 1 microfarad. If a maximum voltage of 100 volts is placed across these circuit parameters, what will be the current flow at any instant after steady state has been established?

A pure resistance across the voltage:

$$i = \frac{v}{R} = \frac{V_m}{R}$$

$$= \tfrac{100}{2} = 50 \text{ amp}$$

The pure inductance across the voltage:

$$i = \frac{1}{\mathfrak{L}} \int v \, dt = \frac{1}{\mathfrak{L}} \int V_m \, dt$$

Mathematically, this condition may be finite or infinite. When applied to the physical system, it will represent a finite flow of current limited only by the resistance that may exist in the wire.

The pure capacity across the voltage:

$$i = C\frac{dv}{dt} = C\frac{d}{dt}V_m$$

$$i = 0$$

These theoretical demonstrations can be proved by laboratory experiments and, as in the series system, to obtain derived end solutions the development depends upon application of mathematical rules to statistical forms.

11. Value of the Derived Forms. The analytic study, which has been made of some possible combinations of the counteraction factors, has introduced new definitions and definite concepts of the probable relationship between two kinds of counteractions (the energy-consuming and non-energy-consuming). The basic relationships which were brought forward in the two developments are the fundamentals in the treatment of electrical circuits and machines. The remainder of the book will deal with the application of these fundamentals and the development of individual concepts for the various parameters of the electrical system.

A clear understanding of the relationship between currents and voltage (where only pure resistance, inductance, and capacity are involved) is of little value in itself but, since most electrical systems are reduced to these factors for analysis, their relationships assume an important position as a tool. The ability to recognize the phase relationships between currents and voltages and to transfer a reference from voltage to current without becoming confused over the relative positions of the cause and effect in time phase is essential.

In power engineering, the parallel system is the most important, because all power transmission of any magnitude is carried over parallel systems. There are two important exceptions to this: one is purely European, where direct current is transmitted at high voltages by the Thury system, and the other is street lighting which is more economically operated as a series system than as a parallel system.

12. Sine Waves and Instantaneous Values. The treatment in this chapter has been with sine waves and their instantaneous values. In the remainder of the book, only the generating vectors (maximum values of the sine waves) which will appear in vector diagrams will be involved, and these will be modified in length by $1/\sqrt{2}$ to enable the diagrams to be drawn to meter reading values without the necessity of converting them to maximum values. Though the remaining work will be with

vector diagrams, it is necessary to keep in mind the fact that voltage and current in a-c systems always depend upon the sine waves that they represent and that the original limitations are carried over into the vector diagrams. A third harmonic may be represented by a revolving vector in the same manner as the fundamental but not on the same vector diagram. This will be evident if a third harmonic wave is added to the fundamental, for the resultant is not another sine wave. Each vector diagram presupposes a constant frequency of the same value for all vectors.

When using instantaneous values, it is not necessary to differentiate between the a-c and d-c circuits but, when dealing with the effective values, the direct current is a scalar quantity, whereas the alternating current is a vector. It is possible to compare the two systems and consider them from the same viewpoint, if the analysis is made from instantaneous values, with the effective and maximum values considered as the reason an instantaneous value exists at a specific time.

PROBLEMS

1–6. A current, $i = 14.14 \sin 377t$, produces the following voltage drops in a series circuit made up of pure elements of resistance, inductance and capacitance: $v_R = 212.1 \sin 377t$, $v_\mathcal{L} = 707.0 \sin [377t + (\pi/2)]$, and $v_C = 989.8 [\sin 377t - (\pi/2)]$. Determine (*a*) resistance, (*b*) inductive reactance, (*c*) capacitive reactance, (*d*) inductance, (*e*) capacitance, and (*f*) power factor for each part, and (*g*) resultant power factor.

2–6. The impedance of a coil is 11.18 ohms. If the inductive reactance is one-half the resistance what is the power factor of the coil?

3–6. A current, $i = 14.14 \sin 377t$ causes a voltage drop of $v_R = 141.4 \sin 377t$ across a resistance with an applied voltage $v_0 = 200 \sin [377t + (\pi/4)]$. The system consists of a pure resistance and reactance. (*a*) Is the system inductive or capacitive? (*b*) What is the value of the inductance or capacitance? (*c*) What power does the system require?

4–6. A coil operating on a 100-volt, 60-cycle system draws 5 amp and 400 watts. At 50 cycles what applied voltage will produce the same power and current flow?

5–6. A series system containing three elements, a resistance of 5 ohms, an inductive reactance of 2 ohms, and a capacitive reactance of 3 ohms draws $i = 14.14 \sin 251.32t$ amp. Determine (*a*) the value of and the expression for the applied voltage, (*b*) the total impedance, (*c*) the system power factor, also the instantaneous (*d*) voltage across the resistance, (*e*) across the inductive reactance, (*f*) across the capacitive reactance, and (*g*) the values of these voltages at time $t = 0.01$ sec.

6–6. A parallel system across $v_0 = 100 \sin 377t$ produces the following currents in the three pure branches in parallel: $i_R = 10 \sin 377t$, $i_\mathcal{L} = 1.25 \sin [377t - (\pi/2)]$, and $i_C = 2.5 \sin [377t + (\pi/2)]$. Determine the values of (*a*) total impedance, (*b*) total admittance, (*c*) resistance, (*d*) inductive reactance, (*e*) capacitive reactance, (*f*) conductance, (*g*) inductive susceptance, (*h*) capacitive susceptance, (*i*) power factor for each part, and also (*j*) power factor for the system.

7–6. A system with a resistance and a capacitive reactance in parallel has an admittance of 2.236 ohms. If the value of the capacitive reactance is one-half that of the resistance what is the resultant power factor of the system?

8–6. When a resistance and a reactance are placed in parallel the resistance branch takes $i_R = 10 \sin 50\pi t$ and the system as a whole $i_0 = 14.14 \sin [50\pi t - (\pi/4)]$ for the current when a voltage $v_0 = 100 \sin 50\pi t$ is applied. Determine (*a*) if the system contains an inductance or a capacity and its value, and (*b*) the power consumed by the parallel system.

9–6. A resistance of 15 ohms and a capacity of 37.9 microfarads are placed in parallel across 100 volts in which the frequency is varied from 60 cycles to 30 cycles. (*a*) What current is drawn at 60 cycles and (*b*) what current is drawn at 30 cycles? (*c*) What is the power drawn at each frequency?

10–6. Across $v_0 = \sin [157t + (\pi/6)]$ are placed a resistance of 10 ohms, an inductive reactance of 10 ohms, and a capacitive reactance of 10 ohms in parallel. Determine (*a*) total impedance, (*b*) total admittance, and (*c*) power factor; also, the expressions for (*d*) the total instantaneous current, (*e*) across the resistance, (*f*) across the inductance, (*g*) across the capacity, and (*h*) the values of these currents at time $t = 0.015$ sec.

11–6. A resistance of 10 ohms, an inductive reactance of 4 ohms, and a capacitive reactance of 2 ohms are connected in series across $v_0 = 100 \sin [80\pi t + (\pi/3)]$. (*a*) What is the current in each parameter when $t = 0.12$ sec? Determine (*b*) the impedance, (*c*) the power factor, and (*d*) the power in every part, and for the system as a whole.

12–6. A resistance of 10 ohms, an inductive reactance of 4 ohms, and a capacitive reactance of 2 ohms are connected in parallel across $v_0 = 100 \sin [80\pi t + (\pi/3)]$. (*a*) What is the current in each parameter when $t = 0.12$ sec? Determine (*b*) the impedance, (*c*) the power factor, and (*d*) the power in every part and for the system as a whole.

13–6. A voltage $v_0 = 100 \sin [377t + (\pi/9)]$ supplies 10 ohms of resistance, 2 ohms of inductive reactance, 4 ohms of capacitive reactance in parallel. Determine (*a*) the current in each part at $t = 0.01$ sec, (*b*) the power consumed by the system.

14–6. Three parallel branches having 10 ohms resistance, 0.1 henry inductance, and 63.69 microfarads of capacitance, respectively, are placed in parallel across $v_0 = 100 \sin 157t$. Determine the current through (*a*) each branch, (*b*) for the whole system.

15–6. A current, $i = 14.14 \sin 157t$, flows through a series system composed of 15 ohms of resistance, 0.1 henry inductance, 63.69 microfarads capacitance. Determine (*a*) the voltage across each parameter, (*b*) the power factor of the system, (*c*) the expression for the applied voltage, (*d*) the power consumed by the system.

CHAPTER 7

SYMBOLIC TREATMENT OF VECTORS: COMPLEX QUANTITIES

1. Reasons for Vector Representation. After a study of the previous chapter on the general circuit, it appears that a solution to a general circuit problem can be obtained only after solving involved equations because of sinusoidal voltage and current expressions. The addition or subtraction of the sine waves of the same frequency representing the currents or voltages of a circuit is difficult and in most cases graphical solutions are used. To obtain a reliable solution of a circuit problem by graphic methods requires considerable time and accuracy. Vector

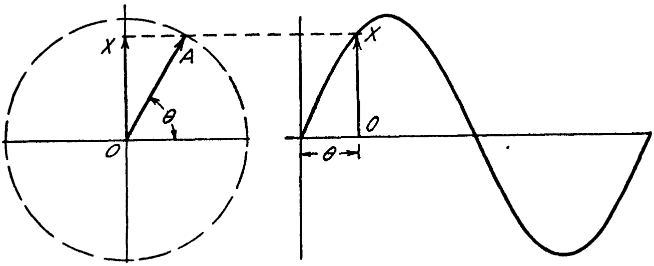


FIG. 1–7. Development of a sine wave from a rotating vector.

representation of the voltages and currents of a circuit and the expressing of these vectors in complex form simplifies the work involved and, at the same time, adds materially to the accuracy of the solution obtained.

It is possible to represent a sine wave in three different ways: (1) graphically (rectangular or polar coordinates); (2) mathematically; (3) vectorially (revolving or stationary vector). The first two methods have been mentioned previously and the difficulties in obtaining solutions indicated. The representation in vector form is the method most generally used in electrical engineering because of its accuracy and the ease with which it can be manipulated.

2. Vector Representation of a Sine Wave. The sine wave may be considered as the development, in rectangular coordinates, of a revolving vector which is constant in magnitude and rotates counterclockwise with a constant angular velocity. In Fig. 1–7, the development, in rectangular coordinates, of the vector OA, as it rotates counterclockwise at a

constant angular velocity, is a sine wave. The sine wave is the envelope of the vertical axis projections at every value of the displacement angle θ, and is equal numerically to $OA \sin \theta$ at every value of the angle θ.

If the sine wave of Fig. 1–7 represents the voltage of an electrical circuit, the revolving vector OA is the maximum value of this voltage.

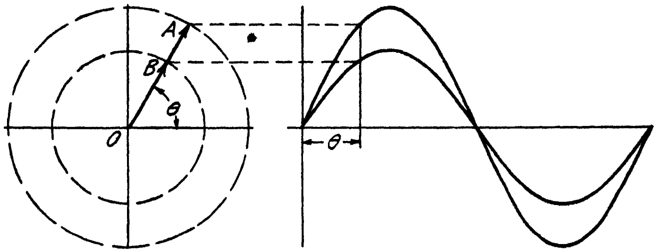


FIG. 2–7. Sine waves produced by the two revolving vectors OA and OB.

The vector OA is the representation, vectorially, of the maximum value of voltage for the circuit. The value of OX is the instantaneous value at some angle θ. Figure 2–7 shows two sine waves in time phase produced by the revolving vectors OA and OB. If the sine waves produced by the vectors OA and OB are added together, the resultant sine wave

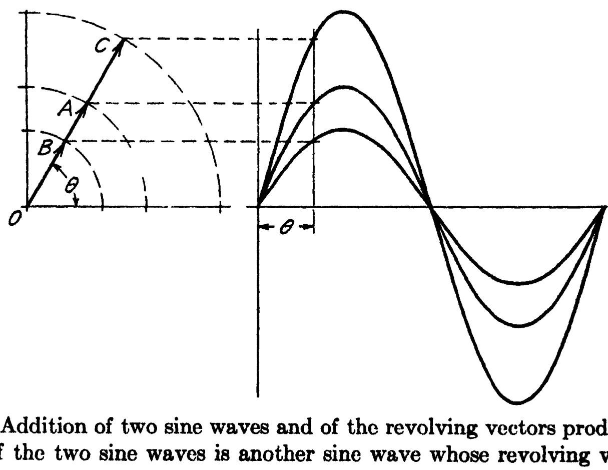


FIG. 3–7. Addition of two sine waves and of the revolving vectors producing them. The sum of the two sine waves is another sine wave whose revolving vector is the vector sum of the two vectors.

is the sum of the two sine waves. The revolving vector, which would produce the same wave, is the sum of the two vectors OA and OB. This summation is shown in Fig. 3–7.

The relationships existing for the sine waves also exist for the revolving vectors which produce them.

3. Scalar and Vector Quantities. A scalar quantity is defined as a physical quantity having only magnitude and being completely expressed by a simple number. This number may be either positive or negative. Examples of scalar quantities expressed by positive numbers are: mass, energy, torque, resistance, time, and money. Examples of quantities expressed by either positive or negative numbers are: temperature and electrostatic charge.

A vector quantity may be defined as a physical quantity having both magnitude and direction. Examples of vector quantities are: force, distance, and velocity. As mentioned previously, there may be two kinds of vectors, (*a*) stationary and (*b*) rotating. The magnitude and direction of a force acting in a given direction can be represented by a stationary vector. A rotating vector is constant in magnitude and rotates at a constant angular velocity. The force exerted by the spoke of a wheel may be considered a rotating vector. Rotating vectors are used in a-c studies and the angle of displacement between two vector quantities is the fixed angle between the rotating vectors representing these quantities.

4. Vector Quantities in Rectangular Coordinates. A plane surface can be divided into four equal quadrants obtained by the intersection of the XX- and YY-axes. These quadrants are numbered as indicated

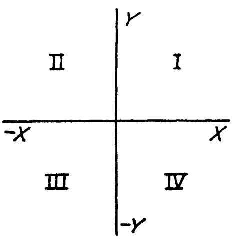


FIG. 4–7. The four quadrants, with counterclockwise rotation assumed.

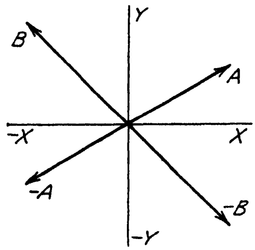


FIG. 5–7. Representation of the vectors A, $-A$, B, $-B$.

in Fig. 4–7, a counterclockwise rotation of vectors being assumed. In the study of trigonometry, the projections of a vector on the XX-axis are called the cosine terms and the projections on the YY-axis are called the sine terms. The values of the cosine terms are positive in the first and fourth quadrants and negative in the second and third. The sine terms are positive in the first and second quadrants and negative in the third and fourth.

In Fig. 5–7, two vectors, $+A$ and $+B$, are shown in the first and second quadrants, respectively. The vectors $-A$ and $-B$ are obvi-

ously equal and opposite to $+A$ and $+B$ and will be in the third and fourth quadrants respectively. The vectors $+A$ and $-A$ are 180° apart, as are the vectors $+B$ and $-B$. The vector $-A$ can be obtained by rotating the vector $+A$ counterclockwise (the order in which the quadrants are numbered) 180° or by multiplying the vector $+A$ by the operator -1. Since the rotation of 180° (two quadrants) is obtained by multiplying by the operator -1, a rotation of 90° (one quadrant) may be obtained by multiplying by the operator $(-1)^{1/2}$ or $\sqrt{-1}$. Therefore, the rotation of a vector 180° counterclockwise can be made in two 90° operations, each operation being obtained by using the operator

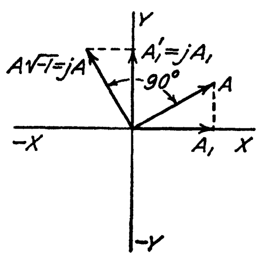


FIG. 6–7. Diagram showing the rotation of a vector (A) when the operator $\sqrt{-1}$ or j is used.

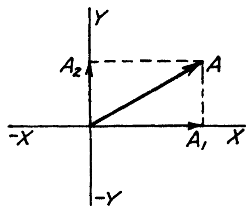


FIG. 7–7. Vector A resolved into two components, A_1, the real axis component, and A_2, the imaginary axis component.

$\sqrt{-1}$. In algebra, the number $\sqrt{-1}$ cannot represent a physical quantity and is called an imaginary number. However, it can be used as an operator for vector quantities and, when used as such, represents an angular rotation of 90° in a counterclockwise direction.

A vector with a component A_1 (Fig. 6–7) along the XX-axis can be rotated 90° counterclockwise by using the operator $\sqrt{-1}$, and it has the same component, A_1, along the YY-axis. The YY-axis is called the axis of imaginaries because the operator $\sqrt{-1}$ is considered an imaginary number. In the field of pure mathematics, the operator $\sqrt{-1}$ is represented by the letter i but, because i is used to denote current, the letter j is used in electrical engineering.

Table I–7 gives the values of the operator j raised to various powers. A vector can be represented by its projections on the XX-axis (axis of reals or cosine terms) and on the YY-axis (axis of imaginaries or of sine terms), and these projections or components, being at right angles to each other, form two sides of a right triangle of which the hypotenuse is the vector itself. In Fig. 7–7, the vector A is reduced into the two components A_1 and A_2, which are at right angles to each other. In this way, it is possible to represent the vector A by a complex algebraic ex-

pression indicating that the vector A is the sum of the two vectors A_1 and A_2, which are at right angles to each other. In order to indicate

TABLE I-7

TABLE OF VALUES OF THE OPERATOR j RAISED TO VARIOUS POWERS

$j = \sqrt{-1}$	$j^6 = -1$
$j^2 = -1$	$j^7 = -\sqrt{-1} = -j$
$j^3 = -\sqrt{-1} = -j$	$j^8 = 1$
$j^4 = +1$	$j^9 = \sqrt{-1} = j$
$j^5 = \sqrt{-1} = j$	$j^{10} = -1$

this relationship in complex algebraic form, the expression is written as

$$\bar{A} = A_1 + jA_2$$

The use of the bar ($\bar{\ }$) above an expression signifies that the relationship is given in complex algebraic form and, when written in this form, it should not be considered as a simple numerical expression. The representation of the vector A as $\bar{A} = A_1 + jA_2$ definitely locates this vector as to magnitude and direction. Its magnitude is equal, numerically, to $(A_1^2 + A_2^2)^{1/2}$. The symbol j is an operator and is used to denote direction and will not appear in the expression indicating the magnitude of a vector. The vectors A, B, C, D, E, and F in Fig. 8-7 can be represented in complex algebraic form as

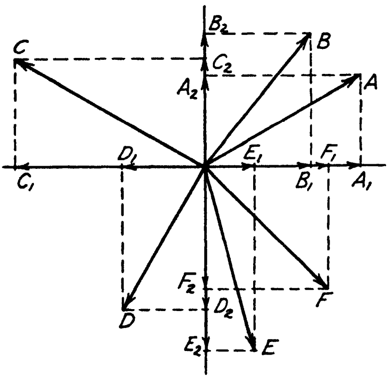


FIG. 8-7. Resolution of vectors into real and imaginary components.

$$\bar{A} = A_1 + jA_2$$
$$\bar{B} = B_1 + jB_2$$
$$\bar{C} = -C_1 + jC_2$$
$$\bar{D} = -D_1 - jD_2$$
$$\bar{E} = E_1 - jE_2$$
$$\bar{F} = F_1 - jF_2$$

5. Addition and Subtraction of Vector Quantities in Complex Form. (*a*) *Addition.* The addition of two vectors gives a resultant vector, the components of which are equal to the sum of the similar components of the vectors being added.

As shown in Fig. 9-7, the sum of vector A and vector B is vector C. Analysis of the magnitude of the XX-axis and YY-axis projections of each vector indicates that $C_1 = A_1 + B_1$ and $C_2 = A_2 + B_2$. In complex algebra, the addition can be indicated as

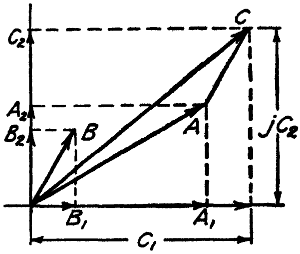


FIG. 9-7. Addition of two vectors. The resultant is the sum of the individual components.

$$\bar{C} = \bar{A} + \bar{B}$$

$$\bar{C} = (A_1 + jA_2) + (B_1 + jB_2)$$

Grouping similar terms,

$$\bar{C} = (A_1 + B_1) + j(A_2 + B_2)$$

or

$$\bar{C} = C_1 + jC_2$$

If any of the components (real or imaginary) are negative, the sign should be indicated in the resultant summation.

Example a. Find the sum of the complex expressions $(12 + j5)$, $(5 + j9)$, and $(-15 - j24)$.

The complex expression representing the sum is

$$\begin{array}{r} 12 + j\,5 \\ 5 + j\,9 \\ -15 - j24 \\ \hline 2 - j10 \end{array}$$

The magnitude of the resultant sum is $(2^2 + 10^2)^{1/2} = \sqrt{104} = 10.2$.

The sum of several complex expressions is obtained by taking separately the sum of the real components and the imaginary components.

(*b*) *Subtraction.* Subtraction is the reverse process of addition and, consequently, in dealing with complex expressions the process is reversed. Referring to Fig. 9-7, if the vectors A and C are given with vector A to be subtracted from vector C, the resultant is vector B. This can be indicated as

$$\bar{C} = C_1 + jC_2 \qquad \bar{A} = A_1 + jA_2$$

$$\bar{B} = \bar{C} - \bar{A} = (C_1 + jC_2) - (A_1 + jA_2)$$

$$\bar{B} = C_1 + jC_2 - A_1 - jA_2$$

$$\bar{B} = (C_1 - A_1) + j(C_2 - A_2)$$

or

$$\bar{B} = B_1 + jB_2$$

To subtract one complex expression from another complex expression, the real components and imaginary components are subtracted separately.

Example b. Subtract $(-7 + j9)$ from $(-3 - j12)$.

$$\begin{array}{r} -3 - j12 \\ -7 + j\ 9 \\ \hline 4 - j21 \end{array}$$

Example c. Find the resultant complex expression for $(4 + j2) - (5 - j9) + (-3 + j12) - (-8 + j20)$.

$$(4 + j2) - (5 - j9) + (-3 + j12) - (-8 + j20) = 4 + j3$$

6. Multiplication and Division of Vector Quantities in Complex Form. (*a*) *Multiplication.* The multiplication of two complex expressions is quite similar to the multiplication of two algebraic expressions, with some special consideration. For example, the product of the two algebraic terms $X = a + b$ and $Y = c + d$ is

$$XY = (a + b)(c + d) = ac + ad + bc + bd$$

Now, consider the product of the complex expressions $\bar{A} = A_1 + jA_2$ and $\bar{B} = B_1 + jB_2$. This product will be a complex expression and can be obtained as

$$\bar{C} = \bar{A} \times \bar{B} = (A_1 + jA_2)(B_1 + jB_2),$$

$$\bar{C} = A_1B_1 + jA_1B_2 + jA_2B_1 + (jA_2)(jB_2),$$

$$\bar{C} = A_1B_1 + jA_1B_2 + jA_2B_1 + j^2A_2B_2 \qquad (\text{but } j^2 = -1)$$

Therefore

$$\bar{C} = A_1B_1 + jA_1B_2 + jA_2B_1 - A_2B_2$$

$$\bar{C} = (A_1B_1 - A_2B_2) + j(A_1B_2 + A_2B_1)$$

or

$$\bar{C} = C_1 + jC_2$$

where

$$C_1 = A_1B_1 - A_2B_2$$

$$C_2 = A_1B_2 + A_2B_1$$

Example d. Multiply $(10 + j15)$ by $(5 - j9)$.

$$\begin{aligned} (10 + j15)\ (5 - j9) &= 50 - j90 + j75 - j^2135 \qquad [-j^2 = -(-1) = 1] \\ &= 50 - j90 + j75 + 135 \\ &= (50 + 135) + j(-90 + 75) \\ &= 185 - j15 \end{aligned}$$

Example e. Find the product of $(5 + j5)\ (3 + j4)\ (1 + j1)$.

$$\begin{aligned}(5 + j5)\ (3 + j4)\ (1 + j1) &= (5 + j5)\ [(3 + j4)\ (1 + j1)] \\ &= (5 + j5)[(3 + j3 + j4 + j^2 4)] \\ &= (5 + j5)\ (-1 + j7) \\ &= (-5 + j35 - j5 + j^2 35) \\ &= -40 + j30\end{aligned}$$

To obtain the product of two or more complex quantities, multiply the individual terms, following the usual rules of algebra, with the exception that the terms containing j to any power are subject to the special rules governing its value numerically. The resultant is then separated into its real and imaginary components.

(*b*) *Division.* The division of two complex quantities is the reverse process of multiplying two complex quantities. First, however, consideration must be given to evaluating the reciprocal of a complex expression. That is, $\dfrac{A_1 + jA_2}{B_1 + jB_2}$ is $(A_1 + jA_2) \times \dfrac{1}{B_1 + jB_2}$, and attention must be given to the expression $\dfrac{1}{B_1 + jB_2}$. From college algebra, the expression $\dfrac{1}{a + b}$ was rationalized as

$$\frac{1}{a + b} = \frac{a - b}{(a + b)(a - b)} = \frac{a - b}{a^2 - b^2} = \frac{a}{a^2 - b^2} - \frac{b}{a^2 - b^2}$$

and $\dfrac{1}{a + b}$ may be written as $M - N$, where $M = \dfrac{a}{a^2 - b^2}$ and $N = \dfrac{b}{a^2 - b^2}$.

In a similar manner the expression $\dfrac{1}{B_1 + jB_2}$ can be rationalized as

$$\frac{1}{B_1 + jB_2} = \frac{B_1 - jB_2}{(B_1 + jB_2)(B_1 - jB_2)} = \frac{B_1 - jB_2}{B_1{}^2 + jB_1B_2 - jB_1B_2 - j^2B_2{}^2}$$

$$= \frac{B_1 - jB_2}{B_1{}^2 + B_2{}^2} = \frac{B_1}{B_1{}^2 + B_2{}^2} - j\frac{B_2}{B_1{}^2 + B_2{}^2}$$

It should be noted that both terms $\dfrac{B_1}{B_1{}^2 + B_2{}^2}$ and $\dfrac{B_2}{B_1{}^2 + B_2{}^2}$ are finite numbers or components and, when expressed in complex form, will have the operator j appearing with the imaginary component.

The division of $(A_1 + jA_2)$ by $(B_1 + jB_2)$ can be performed as

$$\frac{A_1 + jA_2}{B_1 + jB_2} = \frac{(A_1 + jA_2)(B_1 - jB_2)}{(B_1 + jB_2)(B_1 - jB_2)}$$

$$= \frac{A_1B_1 - jA_1B_2 + jA_2B_1 - j^2A_2B_2}{B_1^2 + B_2^2}$$

$$= \frac{(A_1B_1 + A_2B_2) + j(A_2B_1 - A_1B_2)}{B_1^2 + B_2^2}$$

$$= \frac{A_1B_1 + A_2B_2}{B_1^2 + B_2^2} + j\,\frac{A_2B_1 - A_1B_2}{B_1^2 + B_2^2}$$

and the two components of this complex expression are finite numbers.

Example f. Divide $(12 + j2)$ by $(3 - j4)$.

$$\frac{12 + j2}{3 - j4} = \frac{(12 + j2)(3 + j4)}{(3 - j4)(3 + j4)} = \frac{36 + j48 + j6 + j^2 8}{9 + 16}$$

$$= \frac{28 + j54}{25} = \frac{28}{25} + j\,\frac{54}{25} = 1.12 + j2.16$$

Example g. Divide $(-3 + j4)$ by $(1 + j1)$.

$$\frac{-3 + j4}{1 + j1} = \frac{(-3 + j4)(1 - j1)}{(1 + j1)(1 - j1)} = \frac{-3 + j3 + j4 - j^2 4}{1 + 1}$$

$$= \frac{1 + j7}{2} = \frac{1}{2} + j\,\frac{7}{2} = 0.5 + j3.5$$

To divide one complex quantity by another, follow the same rules governing the division of algebraic expressions; namely, rationalize the denominator, multiply the numerators, add like components, and reduce to the simplest complex form.

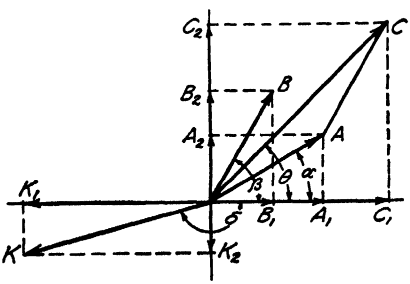


FIG. 10–7. Vector representation in polar coordinates: $A\angle\alpha$, $B\angle\beta$, $C\angle\theta$, $K\angle 360 - \delta$.

7. Polar Equations. A vector may be expressed in polar form as well as in rectangular form. Referring to Fig. 10–7, vector A may be designated by A at an angle α or written in polar form as $A\angle\alpha$. Likewise, vector B may be written as $B\angle\beta$. This notation does not mean that the vector B is the product of the magnitude B and the angle β but simply refers to the position of B displaced an angle β from the

reference axis with the angle taken in the counterclockwise direction. It is possible, however, to express a vector quantity in polar form in two ways. The vector K may be designated in polar form as $K\underline{/360 - \delta}$ or $K\overline{\backslash\delta}$. The latter notation ($\overline{\backslash}$) indicates the angle taken clockwise instead of counterclockwise.

8. Addition and Subtraction of Vector Quantities Expressed in Polar Form. The addition or subtraction of two complex quantities expressed in polar form is accomplished in exactly the same manner as it is when they are expressed in rectangular form. Polar expressions are usually converted to rectangular form before being added or subtracted. The vectors $A\underline{/\alpha}$ and $B\underline{/\beta}$ must be resolved into rectangular coordinates before being added in order to obtain the resultant vector $C\underline{/\theta}$ (Fig. 10–7). $A\underline{/\alpha}$ can be written as A *cjs* (α),* which means that, expressed in rectangular components, $A = A\cos\alpha + jA\sin\alpha$. Likewise, B *cjs* (β) can be written as $B = B\cos\beta + jB\sin\beta$. $A\underline{/\alpha} + B\underline{/\beta}$ then becomes

$$\bar{A} + \bar{B} = (A\cos\alpha + jA\sin\alpha) + (B\cos\beta + jB\sin\beta)$$

$$\bar{A} + \bar{B} = \bar{C} = (A\cos\alpha + B\cos\beta) + j(A\sin\alpha + B\sin\beta)$$

$$= \bar{C} = C\cos\theta + jC\sin\theta$$

and, finally, the resultant vector C can be written in polar form as $C\underline{/\theta}$.

Subtraction is the reverse process of addition and, again, similar components are subtracted from each other, as in rectangular coordinates.

Example h. Add the vector quantities $12\underline{/30°}$ and $15\underline{/45°}$.

$$12\underline{/30°} \text{ is } 12 \; cjs \; 30°$$

or, in rectangular form,

$$12\underline{/30°} = 12\,(\cos 30° + j\sin 30°) = 12(0.866 + j0.5)$$
$$= 10.392 + j6$$
$$15\underline{/45°} = 15(\cos 45° + j\sin 45°) = 15(0.707 + j0.707)$$
$$= 10.605 + j10.605$$
$$(10.392 + j6) + (10.605 + j10.605) = 20.997 + j16.605$$

Magnitude of resultant vector $= (20.997^2 + 16.605^2)^{1/2}$
$= 26.81$

Cosine of the angle of resultant $= \dfrac{20.99}{26.81} = 0.785$

Sine of the angle $= \dfrac{16.6}{26.81} = 0.617$

Resultant $= 26.81(0.785 + j0.617)$
$= 26.81\underline{/38.1°}$

* *cjs* reads cosine plus j sine.

The addition or subtraction of complex quantities expressed in polar form cannot be performed so easily as it can where the complex quantities are expressed in rectangular coordinates.

9. Multiplication and Division of Vector Quantities Expressed in Polar Form. The product of two vector quantities, which are expressed in polar form, is a vector quantity, the magnitude of which is equal to the product of the individual vector magnitudes, expressed at an angle equal to the sum of the angles for the individual vectors.

For example, multiply $A\underline{/\alpha}$ by $B\underline{/\beta}$. The product becomes

$$\begin{aligned}
\bar{A} &= A \text{ cjs } \alpha = A \cos \alpha + jA \sin \alpha \\
\bar{B} &= B \text{ cjs } \beta = B \cos \beta + jB \sin \beta \\
\bar{A} \times \bar{B} &= (A \cos \alpha + jA \sin \alpha)(B \cos \beta + jB \sin \beta) \\
&= AB(\cos \alpha \cos \beta + j \sin \alpha \cos \beta + j \cos \alpha \sin \beta + j^2 \sin \alpha \sin \beta) \\
&= AB[(\cos \alpha \cos \beta - \sin \alpha \sin \beta) + j(\sin \alpha \cos \beta + \cos \alpha \sin \beta)] \\
\bar{A} \times \bar{B} &= AB[\cos (\alpha + \beta) + j \sin (\alpha + \beta)] \\
\bar{A} \times \bar{B} &= AB \text{ cjs } (\alpha + \beta) \\
\bar{A} \times \bar{B} &= AB\underline{/\alpha + \beta}
\end{aligned}$$

Example i. Find the resultant vector when the vectors $25\underline{/100°}$ and $10\underline{/20°}$ are multiplied together.

$$25\underline{/100°} \times 10\underline{/20°} = 25 \times 10\underline{/100° + 20°} = 250\underline{/120°}$$

The division of two vector quantities expressed in polar form is the inverse process of the multiplication procedure. Therefore, the resultant vector, when dividing one polar expression by another, can be obtained by dividing the magnitudes numerically and expressing the resultant angle as the difference of the individual angles.

$$\frac{A\underline{/\alpha}}{B\underline{/\beta}} = \frac{A}{B}\underline{/\alpha - \beta}$$

Example j. Determine the resultant vector, when $25\underline{/100°}$ is divided by $10/20°$.

$$\frac{25\underline{/100°}}{10\underline{/20°}} = 2.5\underline{/100° - 20°} = 2.5\underline{/80°}$$

10. Vector Diagrams in Electrical Engineering. Chapter 6 has shown how the sine waves of voltage and current were displaced in time phase, depending on the circuit opposition. In a circuit containing pure inductive reactance, for example, the current lags the impressed voltage by an angle of 90°. This relationship of voltage and current at the same frequency can be represented by two vectors (rotating vectors), one representing the voltage and the other, the current, with the current

vector lagging in time phase by an angle of 90°. These vectors can be drawn in any quadrant and, as long as the angle of displacement is the same, the vector diagram will always express the same electrical circuit relationship. If these vectors are referred to the corresponding sine waves expressed in rectangular coordinates, the magnitude of the vector

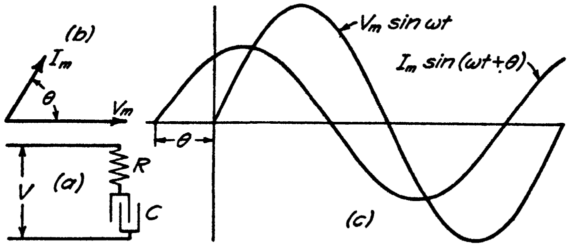


FIG. 11–7. Analysis of a circuit containing R and C.

is the maximum value of the sine wave. For example, a circuit containing resistance and capacitive reactance in series would have its vector diagram drawn as in Fig. 11–7.

The meters used to measure the voltages and currents of a circuit read effective values or maximum values divided by $\sqrt{2}$. Complete circuit information must be available in order to use meter readings in constructing a vector diagram, because the meter readings are only magnitudes. Since meter readings (effective values) are used as a basis of circuit discussion, the effective values can represent the magnitudes

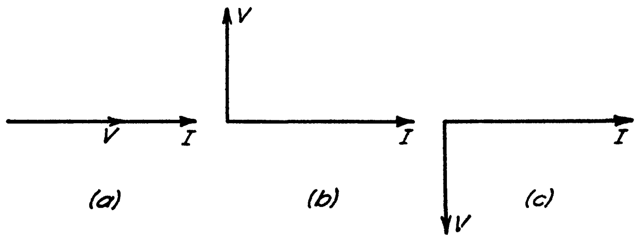


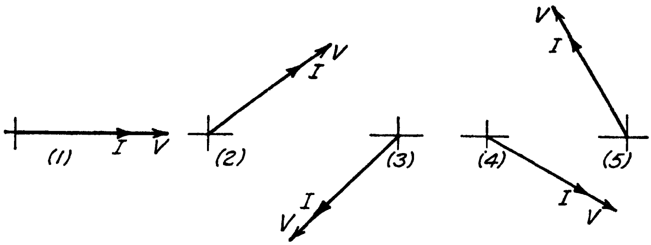
FIG. 12–7. Vector diagrams using effective values: (a) a circuit containing resistance only; (b) a circuit containing inductive reactance only; (c) a circuit containing capacitive reactance only.

of the vectors in the vector diagrams, and the angle of displacement is the angle between the maximum values of the current and voltage waves. The vector diagram of a circuit containing pure resistance can be shown as in Fig. 12–7*a*, where I and V are meter readings and the angle of displacement between I and V is the same as the angle between I_m and V_m. Similarly, Fig. 12–7*b* is the vector diagram of a circuit containing only inductive reactance, and Fig. 12–7*c* is the vector diagram

of a circuit containing only capacitive reactance. In all three diagrams of Fig. 12–7, the vectors are drawn to a definite scale for voltage and current and represent the meter indications. The angle of displacement is dependent upon the circuit counteraction and the information concerning the kind of counteraction must be available before the vector diagram can be constructed. A vector diagram will contain only vectors representing the voltages and currents of a circuit.

11. The Use of Complex Quantities in Electrical Engineering. The use of complex quantities offers a simple way of expressing the positions of the vectors in a vector diagram. Although it is possible to express the vectors in complex quantities, it does not follow that all complex quantities are vectors. The expressing of vectors in complex quantities depends upon the choice of reference in drawing the vectors. Once the vectors are fixed in rectangular coordinates, it is easy to write the complex expressions for them.

As an example, consider the circuit containing a pure resistance of 10 ohms connected to a 100-volt system. The current flowing is $^{100}/_{10}$, or 10 amperes, and will always be in time phase with the impressed voltage. Corresponding to every position selected for the voltage vector, there will be a current vector in time phase with it, and complex expressions can be written for both voltage and current vectors in any position selected. Five different positions are selected in this example for the voltage and current vectors and the complex expressions are given for each position. Figure 13–7 shows the vector diagrams of the



(1) $\bar{V} = 100 + j0$
$\bar{I} = 10 + j0$

(2) $\bar{V} = 80 + j60$
$\bar{I} = 8 + j6$

(3) $\bar{V} = -70.7 - j70.7$
$\bar{I} = -7.07 - j7.07$

(4) $\bar{V} = 86.66 - j50$
$\bar{I} = 8.66 - j5$

(5) $\bar{V} = -50 + j86.66$
$\bar{I} = -5 + j8.66$

FIG. 13–7. Vector diagrams.

five positions. In each of the positions the magnitude $V = 100$, and $I = 10$, and the vectors drawn from the complex expressions indicate the voltage and current in time phase. The kind of impedance determines the relationship between its current and voltage vectors.

Example k. If a sinusoidal voltage with an effective value of 100 volts is impressed upon a pure inductive reactance of 10 ohms, determine the complex expression for the current if $\bar{V} = -50 + j86.66$.

The current is 100/10 = 10 amp and will lag the voltage by 90°. A vector rotated 90° clockwise (lag) must be multiplied by the operator $-j$. The complex expression for the current becomes

$$\frac{-50 + j86.66}{10} \times (-j)$$

and is

$$(-5 + j8.666)(-j) = 8.666 + j5$$

Therefore,

$$\bar{V} = -50 + j86.66$$

$$\bar{I} = 8.666 + j5$$

and the vector diagram is that shown in Fig. 14-7.

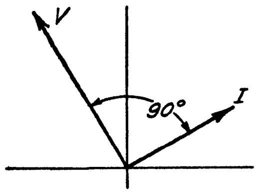


FIG. 14-7. Vector diagram for Example *k*.

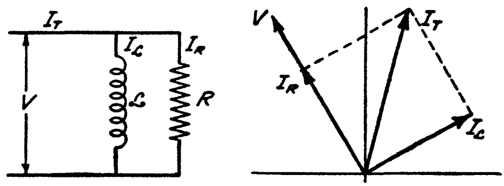


FIG. 15-7. Vector diagram for Example *l*.

Example l. If a sinusoidal voltage of 100 volts is impressed upon a 10-ohm resistance and a 10-ohm inductive reactance in parallel, find the current flowing in each branch and the total current. If $\bar{V} = -50 + j86.6$, what is the complex expression for each current?

If $V = 100$ volts, the current through each impedance is 100/10 = 10 amp. The current through the resistance is in phase with the voltage and the current through the inductive reactance lags the voltage by 90°.

$$\bar{I}_R = \frac{-50 + j86.6}{10} = -5 + j8.66$$

$$\bar{I}_L = \frac{-50 + j86.66}{10} \times (-j) = (-5 + j8.66)(-j)$$
$$= 8.66 + j5$$

$$\bar{I}_T = \bar{I}_R + \bar{I}_L = (-5 + j8.66) + (8.66 + j5)$$
$$= 3.66 + j13.66$$

$$I_T = 14.14 \text{ amp.}$$

Figure 15-7 shows the vector diagram.

In the next chapters, it will be shown that the oppositions of the circuit may be expressed in complex numbers and that the solution of electric circuits can be obtained by using complex numbers to express the cause, opposition, and effect.

PROBLEMS

1–7. Evaluate

$$[(5 + j9) - (4 + j3) + (-7 + j2)] \div [(-2 + j4) - (-5 - j6)]$$

2–7.

$$V_T = \frac{(50 + j50) - (25 - j25)\,(4 - j3)}{(10 + j10) + (5 - j2)}$$

What is the magnitude of V_T?

3–7. Evaluate

$$\frac{1}{3 + j4} + \frac{1}{4 - j3} + \frac{1}{5 + j0}$$

4–7. $\bar{V}_{AB} = -50 + j50$, $\bar{V}_{CD} = 150 + j50$, and $\bar{V}_{EF} = 0 + j100$. (*a*) What is the resultant of $\bar{V}_{AB} + \bar{V}_{CD} + \bar{V}_{FE}$? (*b*) What is the resultant of $\bar{V}_{BA} + \bar{V}_{FE} - \bar{V}_{DC}$?

5–7. A voltage, $\bar{V} = 50 + j50$, is impressed upon a resistance of 5 ohms and an inductive reactance of 10 ohms connected in parallel. What are the complex expressions for the currents through each opposition?

6–7. A resistance of 10 ohms and a capacitive reactance of 5 ohms are connected in series and the current flowing has the expression $\bar{I} = -8 - j6$. What is the expression for the voltage across each opposition and the total voltage?

7–7. A voltage, $\bar{V} = 100\underline{/30^\circ}$, is impressed upon a circuit and the current flowing is $\bar{I} = 20\overline{/30^\circ}$. What is the value of $\bar{Z}$ in rectangular complex form?

8–7. A voltage, $\bar{V} = 200\underline{/30^\circ}$, is impressed upon an impedance, $\bar{Z} = 10\overline{/45^\circ}$. What is $\bar{I}$?

9–7. A condenser of 53 microfarads, an inductance of 0.106 henry and a resistance of 50 ohms are connected in parallel to a 60-cycle voltage, $\bar{V} = 80 + j60$. Determine the complex expression for the current in each opposition and the total current.

10–7. Two sources of voltage, $\bar{V}_{AB} = 90 + j10$ and $\bar{V}_{BC} = 10 + j90$, are connected in series. The voltage $\bar{V}_{AC}$ is connected across a resistance of 10 ohms and an inductive reactance of 10 ohms connected in parallel. What will be the reading of an ammeter connected to measure the total current?

11–7. Two impedances, $\bar{Z}_1 = 5 + j5$ and $\bar{Z}_2 = 4 - j3$, are connected in parallel. What is the equivalent impedance of this system?

12–7. Evaluate

$$\frac{(141.4\underline{/45^\circ}) + (80 + j60)}{(10 + j10) - 5\overline{/90^\circ}}$$

Give the answer in complex rectangular form.

13–7. A voltage vector of 100 volts is drawn at an angle of 45° to the XX'-axis in the first quadrant. The current vector of 10 amp is drawn at an angle of 30° with the YY'-axis in the second quadrant. What is the complex expression for the impedance?

14-7. The current taken by one branch of a parallel circuit is 10 amp and it lags the voltage by 45°. The current taken by the other branch is 10 amp and it leads the voltage by 90°. What is the magnitude of the total current and what is its position relative to the voltage?

15-7. To what magnitude should the current in the branch which leads the voltage in Prob. 14 be changed to give a minimum total current? What is its position relative to the voltage?

CHAPTER 8

CIRCUIT PARAMETERS IN SERIES

1. General Circuit. The discussions in Chapters 3, 4, and 5 considered the characteristics of the three kinds of opposition or counteraction to the flow of current. Each of these oppositions (R, $\mathcal{L}$, and C) reacts differently to the flow of current, and the combined oppositions in series will be proportional to the amount of each opposition present in the electrical circuit.

In Chapter 6, it was shown that, if resistance and reactance are connected in series, the sinusoidal voltage $v = V_m \sin(\omega t \pm \theta)$, necessary to cause a current $i = I_m \sin \omega t$ to flow, may be determined from the expression

$$v = v_R + v_{\mathcal{L}} + v_C$$

The circuit diagram for this relationship is shown in Fig. 1–6. When the voltage components caused by R, $\mathcal{L}$, and C are expressed in effective values, all voltages must be added vectorially and the relation becomes $\bar{V} = \bar{V}_R + \bar{V}_{\mathcal{L}} + \bar{V}_C$. The total voltage across an electric circuit is the vector sum of the voltages across the various component parts.

2. Resistance and Inductive Reactance in Series. In the circuit containing R and $X_{\mathcal{L}}$ in series, the total voltage impressed upon the circuit consists of two components: (1) the component to counteract the effect of resistance and (2) the component to counteract the effect of the inductance.

$$\bar{V} = \bar{V}_R + \bar{V}_{\mathcal{L}}$$

where

$$V_R = \frac{I_m R}{\sqrt{2}}, \quad \bar{V}_R = \bar{I}R$$

$$V_{\mathcal{L}} = \frac{I_m \mathcal{L}\omega}{\sqrt{2}}, \quad \bar{V}_{\mathcal{L}} = j\bar{I}X_{\mathcal{L}}$$

Figure 1–8 shows the circuit and vector diagram of a series circuit. The current vector is the reference vector. It is common practice, when drawing a vector diagram for an electrical circuit, to choose as the reference vector the current or voltage vector which is common to the others.

The vector diagram of Fig. 1–8 is drawn with the current vector as the reference because there is only one current vector and it is common to the three voltage vectors.

The total impedance Z of this series circuit is equal to $(R^2 + X_{\mathcal{L}}^2)^{1/2}$ and the total voltage $V_0 = IZ$. The impedance is expressed in ohms,

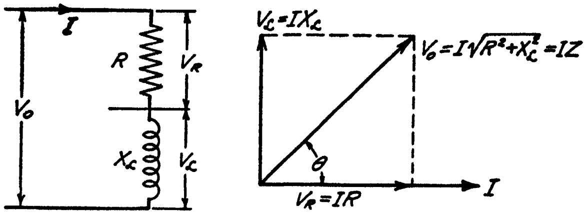


FIG. 1–8. A circuit diagram and vector diagram of a series circuit containing resistance and inductance.

the unit used for resistance and reactance. When the values of current and opposition for the voltages are substituted in the expression $\bar{V}_0 = \bar{V}_R + \bar{V}_{\mathcal{L}}$, the relationship becomes

$$\bar{I}\bar{Z} = \bar{I}\bar{R} + \bar{I}\bar{X}_{\mathcal{L}}$$

The total impedance of the series circuit Z is the vector sum of two impedances in series, one consisting of pure resistance and the other consisting of pure inductive reactance.

In the expression $\bar{V}_0 = \bar{I}\bar{Z}$, the voltage and current can be expressed in complex quantities and, likewise, the impedance can be expressed by a complex quantity. This can be shown also from the relation $Z = V_0/I$. If V and I are expressed in complex form and the expression is reduced to its simplest form, the value of Z will be in complex form. This does not mean, however, that the impedance is a vector but that it is to be considered a complex operator to be used with the current (expressed in complex form) to give the voltage (expressed in complex form).

The vector diagram of Fig. 1–8 shows $V_R = IR$ written in complex quantities: $\bar{V}_R = V_R + j0$ and $\bar{I} = I + j0$. The impedance for this part of the circuit is pure resistance and is equal to

$$\frac{V_R + j0}{I + j0} = \frac{IR + j0}{I + j0} = R + j0$$

An impedance containing only pure resistance can be expressed in complex form as $R + j0$.

The relationship of $V_L = IX_L$ exists and in complex form it is written: $\bar{V}_L = 0 + jV_L$ and $\bar{I} = I + j0$. The impedance for this part of the circuit is pure inductive reactance and can be expressed as

$$\frac{0 + jV_L}{I + j0} = \frac{0 + jIX_L}{I + j0} = 0 + jX_L$$

Thus, an impedance containing only pure inductive reactance can be expressed as $0 + jX_L$.

The total voltage

$$\bar{V}_0 = \bar{V}_R + \bar{V}_L$$

By substitution

$$\bar{I}\bar{Z} = (IR + j0) + (0 + jIX_L)$$

$$\bar{I}\bar{Z} = (IR + jIX_L)$$

and

$$\bar{Z} = R + jX_L$$

For a circuit containing resistance and inductive reactance in series, the total voltage V_0 may be represented as

$$\bar{V}_0 = \bar{I}\bar{Z} = \bar{I}(R + jX_L)$$

where R and X_L are numerical values representing the resistance and reactance in ohms. The angle θ between the voltage vector $\bar{V}_0$ and the current vector $\bar{I}$ is called the power factor angle.

The value of

$$\cos\theta = \frac{IR}{IZ} = \frac{R}{Z} = \frac{R}{\sqrt{R^2 + X_L{}^2}}$$

and the value of

$$\sin\theta = \frac{IX_L}{IZ} = \frac{X_L}{Z} = \frac{X_L}{\sqrt{R^2 + X_L{}^2}}$$

Depending upon the circuit constants, the power factor angle can vary between the limits of 0° and 90°. Likewise, the values of $\cos\theta$ and $\sin\theta$ will vary between the limits of 0 and 1. In Chapter 11, it is shown that the cosine and sine functions of the power factor angle are always considered in power and reactive volt-ampere measurements.

Example a. A series circuit containing a resistance of 10 ohms and an inductance of 0.1 henry is connected to a 220-volt, 60-cycle supply. Find the

current, the voltage across the resistance, the voltage across the reactance, and the power factor angle.

$$X_L = 2\pi(60)(0.1) = 37.7 \text{ ohms}$$

$$\bar{Z} = 10 + j37.7$$

$$Z = \sqrt{10^2 + 37.7^2} = 39 \text{ ohms}$$

$$\text{Power factor} = \cos\theta = \tfrac{10}{39} = 0.256$$

$$R = 10 \text{ ohms}$$

$$I = \tfrac{220}{39} = 5.64 \text{ amp}$$

$$V_R = 5.64 \times 10 = 56.4 \text{ volts}$$

$$V_L = 5.64 \times 37.7 = 212.63 \text{ volts}$$

$$\theta = 75.2°$$

Example b. Find the current if 220 volts direct current is used.

If direct current is used, the value of $X_L = 0$. The value of the total opposition is 10 ohms.

$$I = \tfrac{220}{10} = 22 \text{ amp}$$

$$V_R = 220 \text{ volts}$$

$$\cos\theta = \frac{R}{\sqrt{R^2 + X^2}} = \frac{R}{\sqrt{R^2}} = \frac{10}{10} = 1$$

3. Resistance and Capacitive Reactance in Series. The total voltage impressed upon resistance and capacitive reactance in series consists of two components: (1) the component of voltage to counteract the effect of resistance and (2) the component of voltage to counteract the effect of capacitive reactance.

$$\bar{V}_0 = \bar{V}_R + \bar{V}_C$$

where

$$V_R = \frac{I_m R}{\sqrt{2}}, \quad \bar{V}_R = \bar{I}R$$

and

$$V_C = \frac{I_m}{\omega C\sqrt{2}}, \quad \bar{V}_C = -j\bar{I}X_C$$

Figure 2–8 shows the circuit and vector diagram with the current vector as a reference. The voltage, V_0, is equal numerically to

$$\sqrt{(IR)^2 + (IX_C)^2}$$

and is also equal to the product of the current I and the opposition $Z = \sqrt{R^2 + X_C^2}$. The voltage across the capacitive reactance is at

90° to the current and behind the current vector in time phase. The phase relationship between voltage and current for capacitive reactance is opposite to the phase relationship for inductive reactance, and the counteraction caused by capacity is written in complex form as $0 - jX_C$.

The total opposition of the circuit (Fig. 2–8) can be expressed in complex form as

$$(R + j0) + (0 - jX_C) = R - jX_C = \bar{Z}$$

and the total voltage as

$$\bar{V}_0 = \bar{I}\bar{R} + \bar{I}\bar{X}_C = \bar{I}\bar{Z} = \bar{I}(R - jX_C)$$

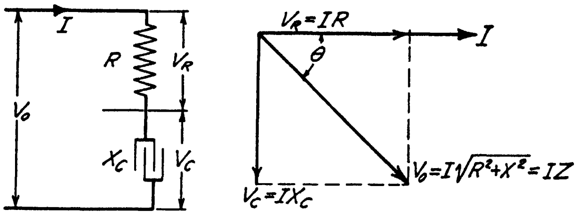


FIG. 2–8. A circuit and vector diagram of a series circuit containing resistance and capacitance.

As in the series circuit containing R and X_L, expressing Z in complex form means that it is to be used as a complex operator to establish the relationship between the circuit voltages and currents. The angle θ between the voltage V_0 and the current I is the power factor angle. The value of $\cos\theta$ is

$$\frac{IR}{IZ} = \frac{R}{Z} = \frac{R}{\sqrt{R^2 + X_C^2}}$$

Example c. A circuit containing a resistance of 30 ohms and a capacitive reactance of 40 ohms is connected to a 100-volt, 60-cycle supply. Determine the current, the voltage across the resistance, the voltage across the capacity, and the power factor angle.

$$Z = \sqrt{R^2 + X_C^2} = \sqrt{30^2 + 40^2} = 50 \text{ ohms}$$

$$I = \tfrac{100}{50} = 2 \text{ amp}$$

$$V_R = 2 \times 30 = 60 \text{ volts}$$

$$V_C = 2 \times 40 = 80 \text{ volts}$$

$$\cos\theta = \tfrac{30}{50} = 0.6$$

$$\theta = 53.1°$$

Example d. If 100 volts of direct current are impressed on the above circuit, how much current is used?

Direct current will flow until the condenser is charged; then the current drops to zero, $I = 0$.

4. Summary of Resistance and Inductive Reactance in Series, Resistance and Capacitive Reactance in Series. If a counterclockwise angular displacement is accepted as positive in sign, the sign of the imaginary term of the complex expression for an impedance containing only an $X_{\mathcal{L}}$ is positive. If the impedance contains only an X_C, the sign of the imaginary term of the complex expression is negative. When the complex form of the impedance is used as an operator with the current, the resultant product is the voltage in correct magnitude and direction.

Therefore, a series circuit containing resistance and inductive reactance will have the total counteraction or impedance (written in complex form)

$$\bar{Z} = R + jX_{\mathcal{L}}$$

and a circuit containing resistance and capacitive reactance will have the impedance (written in complex form)

$$\bar{Z} = R - jX_C$$

5. Resistance, Inductive Reactance, and Capacitive Reactance in Series. In a series circuit, containing R, $X_{\mathcal{L}}$, and X_C in series, the total voltage is composed of three separate voltage components—one for each of the counteractions—and the total voltage may be written

$$\bar{V}_0 = \bar{V}_R + \bar{V}_{\mathcal{L}} + \bar{V}_C$$

The vector diagram of Fig. 3–8 shows that the voltage across the inductive reactance and the voltage across the capacitive reactance are in

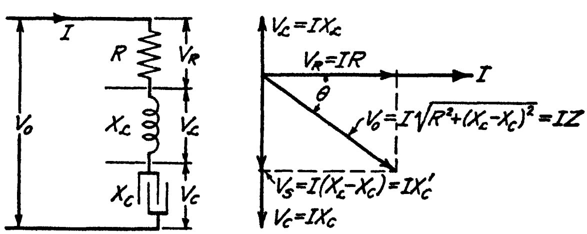


FIG. 3–8. Circuit and vector diagrams of a series circuit containing resistance, inductance, and capacitance.

opposition and that the resultant voltage across the two is their vector sum. In this diagram, IX_C is larger than $IX_{\mathcal{L}}$; therefore, when $V_{\mathcal{L}}$ is added vectorially to V_C, the resultant is V_S. This may be indicated in

vector relationship as $\bar{V}_S = \bar{V}_{\mathcal{L}} + \bar{V}_C$. Since $X_C > X_{\mathcal{L}}$ in this example, the resultant reactance becomes X'_C, which is equal to $X_{\mathcal{L}} - X_C$. The voltage across the resultant reactance may be written

$$V_S = I(X_{\mathcal{L}} - X_C) = IX'_C$$

When the total voltage V_0 and current I are considered (the current leading by an angle θ), the entire circuit presents itself as a simple series circuit containing only a resistance R and a capacitive reactance X'_C. This means that the circuit as shown in Fig. 3–8 reduces to the circuit of Fig. 2–8, when $X_C > X_{\mathcal{L}}$. If the inductive reactance is greater than the capacitive reactance, the reverse is true and the circuit of Fig. 3–8 reduces to that of Fig. 1–8.

Since the total voltage impressed on a circuit is equal to the vector sum of all voltages in the circuit:

$$\bar{V}_0 = \bar{I}\bar{R} + \bar{I}\bar{X}_{\mathcal{L}} + \bar{I}\bar{X}_C$$

and the absolute value

$$V_0 = \sqrt{(IR)^2 + (IX_{\mathcal{L}} - IX_C)^2}$$

$$V_0 = I\sqrt{R^2 + (X_{\mathcal{L}} - X_C)^2}$$

$$V_0 = IZ$$

$$Z = \sqrt{R^2 + (X_{\mathcal{L}} - X_C)^2}$$

The total impedance Z can be expressed in complex form as

$$\bar{Z} = R \pm jX_0$$

where

$$X_0 = X_{\mathcal{L}} - X_C$$

If $X_C > X_{\mathcal{L}}$, the expression is $R - jX_0$; if $X_{\mathcal{L}} > X_C$, the expression is $R + jX_0$.

In every instance the value of $\cos\theta$ is always

$$\frac{R}{Z} \quad \text{or} \quad \frac{R}{\sqrt{R^2 + (X_{\mathcal{L}} - X_C)^2}}$$

Example e. A series circuit containing $R = 10$ ohms, $X_{\mathcal{L}} = 20$ ohms, and $X_C = 10$ ohms has a current of 10 amp. If the complex expression for the current is $10 + j0$, give in complex form V_R, $V_{\mathcal{L}}$, V_C, and V_0. Find the magnitude of each voltage component and power factor angle.

$$\bar{Z} = R \pm jX_0$$

$$X_{\mathcal{L}} > X_C$$

Then

$$\bar{Z} = 10 + j10$$

$$\bar{V}_R = \bar{I}\bar{R} = (10 + j0)(10 + j0) = 100 + j0$$

$$\bar{V}_{\mathcal{L}} = \bar{I}\bar{X}_{\mathcal{L}} = (10 + j0)(0 + j20) = 0 + j200$$

$$\bar{V}_C = \bar{I}\bar{X}_C = (10 + j0)(0 - j10) = 0 - j100$$

$$\bar{V}_0 = \bar{I}\bar{Z} = (10 + j0)(10 + j10) = 100 + j100$$

$$V_R = 100 \text{ volts}$$

$$V_{\mathcal{L}} = 200 \text{ volts}$$

$$V_C = 100 \text{ volts}$$

$$V_0 = 141.4 \text{ volts}$$

$$\cos\theta = \frac{R}{Z} = \frac{10}{14.14} = 0.707$$

$$\theta = 45°$$

In the example given, the voltage across the inductive reactance is greater than the line voltage but, when considered vectorially with V_C and V_R, the resultant gives the line voltage in magnitude and direction. This condition would be impossible in a d-c system, because the voltage components are in the same straight line and the sum, vectorially, is the algebraic sum.

The resultant voltage $(\bar{I}\bar{X}_{\mathcal{L}} + \bar{I}\bar{X}_C)$ will always be less than the line voltage because it is added vectorially to $\bar{I}\bar{R}$ to give the total voltage $\bar{I}\bar{Z}$; $\bar{I}\bar{X}_{\mathcal{L}}$, and $\bar{I}\bar{X}_C$, however, may both be greater than the line voltage but their resultant voltage, when added vectorially to $\bar{I}\bar{R}$, must give the total voltage $\bar{I}\bar{Z}$.

6. Resonance in a Series Circuit. The discussion in Sec. 5 considers only the examples in which $X_{\mathcal{L}} > X_C$ and $X_C > X_{\mathcal{L}}$. There is one other possible arrangement for the series circuit, namely, $X_{\mathcal{L}} = X_C$. If $X_{\mathcal{L}} = X_C$ in the equation,

$$V_0 = I\sqrt{R^2 + (X_{\mathcal{L}} - X_C)^2}$$

$$V_0 = I\sqrt{R^2} = IR$$

and

$$I = \frac{V_0}{R}$$

This means that the current in the series circuit, when $X_{\mathcal{L}} = X_C$, is limited in magnitude by the resistance only. When this condition $(X_{\mathcal{L}} = X_C)$ exists in a series circuit, the circuit is in electrical resonance and the current is a maximum value.

Example f. A series circuit containing $R = 50$ ohms, $X_{\mathcal{L}} = 60$ ohms, and $X_C = 60$ ohms is connected to a 100-volt, a-c system. Determine the magnitude of the current flowing and the voltage across each part of the circuit.

$$\bar{Z} = R + j(X_{\mathcal{L}} - X_C) \qquad I = \tfrac{100}{50} = 2 \text{ amp}$$

$$\bar{Z} = 50 + j(60 - 60) \qquad V_R = 50 \times 2 = 100 \text{ volts}$$

$$\bar{Z} = 50 + j0 \qquad V_{\mathcal{L}} = 60 \times 2 = 120 \text{ volts}$$

$$Z = R = 50 \text{ ohms} \qquad V_C = 60 \times 2 = 120 \text{ volts}$$

For resonance in a series circuit,

$$X_{\mathcal{L}} = X_C$$

$$2\pi f \mathcal{L} = \frac{1}{2\pi f C}$$

or

$$2\pi f \mathcal{L} - \frac{1}{2\pi f C} = 0$$

Solving for frequency,

$$(2\pi)^2 f^2 \mathcal{L} C - 1 = 0$$

$$f^2 = \frac{1}{(2\pi)^2 \mathcal{L} C}$$

and

$$f = \frac{1}{2\pi\sqrt{\mathcal{L} C}}$$

In an ideal series system containing R, $\mathcal{L}$, and C as lumped constants, there is only one frequency at which $X_{\mathcal{L}} = X_C$. The circuit will be in resonance at this frequency (Fig. 4–8), and this value of frequency is

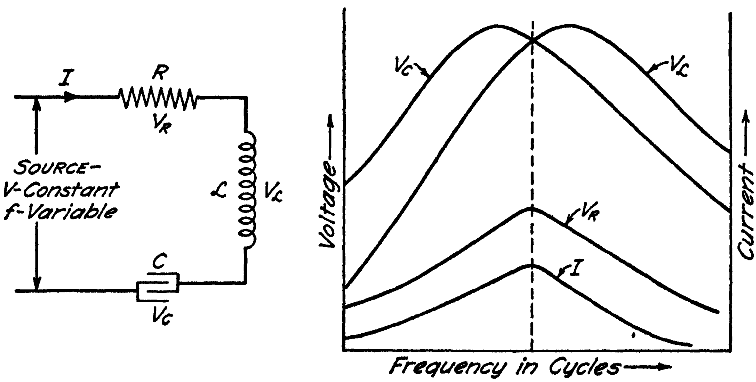


FIG. 4–8. Resonance in an ideal series circuit. The voltage V_R equals V_O, and the current is maximum when V_C equals $V_{\mathcal{L}}$.

often called the natural frequency of the circuit. At this natural frequency the current of the circuit is a maximum and the behavior of the total circuit is the same as that of a circuit containing only resistance. Figure 5-8 shows the actual conditions of a series circuit. Since the

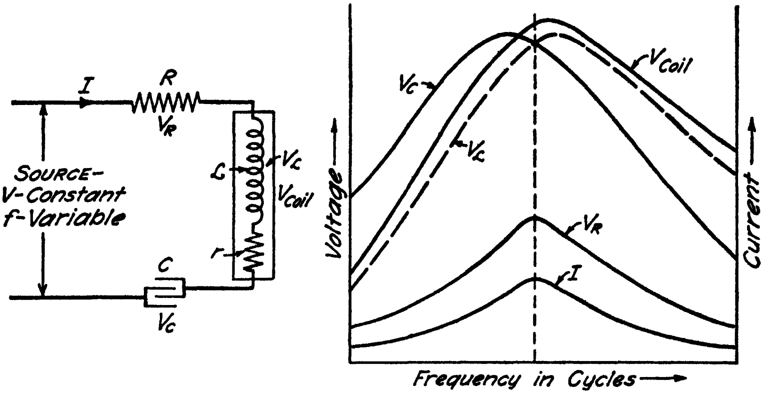


FIG. 5-8. Resonance in a true series circuit. The voltage V_O equals $V_R + V_{R\,coil}$ at resonance. The current is not a maximum when V_C equals V_{coil}.

inductance coil contains resistance, the maximum values of V_{coil} and V_C do not come at the same frequency. The variation from the ideal conditions of Fig. 4-8 is indicated.

An inspection of the factors involved in the expression

$$f = \frac{1}{2\pi\sqrt{\mathcal{L}C}}$$

shows that, if the frequency is fixed, there are an infinite number of combinations of $\mathcal{L}$ and C which will satisfy resonance at that frequency.

In the field of radio, the values of $\mathcal{L}$ and C are adjusted to give maximum signal strength at the broadcast-station frequency. This adjusting of the capacity and inductance is called "tuning" the circuit. At the present time, most radio sets have $\mathcal{L}$ constant and resonance is obtained by varying the condensers, which means varying the value of capacity.

Example g. A series circuit contains $R = 10$ ohms, $\mathcal{L} = 0.1$ henry, and $C = 50$ microfarads. At what frequency is this circuit resonant?

$$f = \frac{1}{2\pi(0.1 \times 50 \times 10^{-6})^{1/2}} = \frac{10^3}{2\pi\sqrt{5}}$$

$$f = 71.3 \text{ cycles}$$

Example h. A series circuit containing $R = 10$ ohms and $X_C = 20$ ohms is connected to a 100-volt, 25-cycle supply. What value of inductance must be connected in series with this circuit to make the circuit resonant? How much current flows before and after the inductance is connected in the circuit?

To establish resonance,

$$X_{\mathcal{L}} = X_C = 20 \text{ ohms}$$

$$\mathcal{L} = \frac{20}{50\pi} = 0.1273 \text{ henry}$$

(Before) $$I = \frac{100}{\sqrt{10^2 + 20^2}} = \frac{100}{\sqrt{500}} = \frac{100}{22.36} = 4.48 \text{ amp}$$

(After) $$I = \tfrac{100}{10} = 10 \text{ amp}$$

7. Summary: Series Circuits.

(1) Voltages add vectorially: $\overline{V}_0 = \overline{V}_1 + \overline{V}_2 + \overline{V}_3 + \overline{V}_N$
(2) Impedances add vectorially: $\overline{Z}_T = \overline{Z}_1 + \overline{Z}_2 + \overline{Z}_3 + \overline{Z}_N$
(3) Resistances add numerically: $R_T = R_1 + R_2 + R_3 + R_N$
(4) Reactances add algebraically: $X_T = \pm X_1 \pm X_2 \pm X_3 \pm X_N$
(5) The current vector is generally chosen as the reference for drawing vector diagrams.
(6) $$Z = \sqrt{R^2 + (X_{\mathcal{L}} - X_C)^2}$$
(7) Power factor: $\cos\theta = \dfrac{R}{Z}$

8. Locus Diagrams. In all the previous discussions, it has been assumed that the parameters are to be held constant and that each solution is for one specific value. Increments of the system parameters may be added or removed from the system, or there may be a change in the voltage, current, or frequency. Each of these factors may change individually or several may change at the same time.

A graphical representation of the complete analysis of these changes is in the form of a locus diagram. The use of such a diagram to visualize these changes is the common practice of the engineer, because it enables him to comprehend numerous changes without the necessity of complicated algebraic developments.

The graphical representations employed in the usual engineering problems are not sufficient for demonstrating the complete analysis of circuit problems solved by Kirchhoff's Laws in symbolic form. The locus diagram is a graphical form that lends itself to this type of problem, and the loci are constructed by the use of the principles of analytical geometry.

The number of problems that may be studied by this method is limitless. The usual forms of the loci in electrical engineering are either straight lines, circles, or parabolas; however, many complex curves may be obtained. The usual type of locus results from a change of one of the parameters, the voltage, or the current, or a combination of these changes.

9. Circuits with Resistance and Inductive Reactance. Specific treatment will be given the combination of resistance and inductive reactance in series, because these represent the class of problems used by the

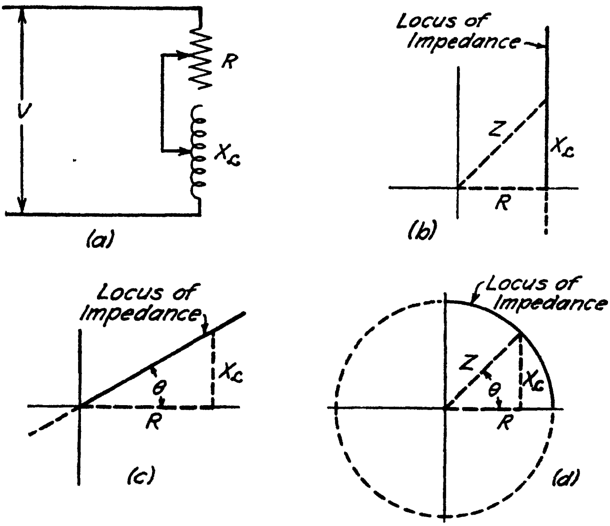


FIG. 6–8. Locus diagrams for impedance.

power engineer in machines and circuits. For the energy engineer in the field of communication, just as much stress must be placed on the capacitive circuit.

Figure 6–8*a* represents a circuit in which both parameters (resistance and inductive reactance) have provisions for adjustment in value. Figure 6–8*b* shows the locus of the impedance for constant R and $\mathcal{L}$ with variable frequency. For constant power factor the locus diagram is shown in Fig. 6–8*c*. The angle θ is constant. Figure 6–8*d* shows the locus diagram of the impedance for variable power factor with the scalar value of the impedance Z constant. These parameter loci are classified under the limitations set by the following equations.

$$\bar{Z} = R + j2\pi f\mathcal{L}$$

$$\tan\theta = \frac{2\pi f\mathcal{L}}{R}$$

In the locus diagrams, the dotted portion of the locus is of no physical importance in the given discussion since resistance is neither quantitatively negative nor can it be negative in a relative sense. Also, those problems which fall under capacitive reactance are not considered.

Case I (Fig. 6–8*b*). The resistance (R) and the inductance ($\mathcal{L}$) are constant; the frequency is varied. Determine the locus diagram for the impedance values.

$$\bar{Z} = R + j2\pi f\mathcal{L}$$

$$\bar{Z} = R + jkf$$

$$z + jz' = R + jkf$$

Therefore

$$z = R$$

$$z' = kf$$

These are a set of parametric equations for Z which describe completely the possible loci. When the parameter (R) is constant in value, the locus given by the general form $x = R$ is shown in Fig. 6–8*b*.

Case II (Fig. 6–8*c*). A variable impedance with a constant power factor. Determine the locus diagram for the impedance values.

$$\bar{Z} = R + jX_{\mathcal{L}}$$

$$\tan\theta = \frac{X_{\mathcal{L}}}{R} = k$$

$$z + jz' = R + jX_{\mathcal{L}}$$

$$z = R$$

$$z' = X_{\mathcal{L}}$$

$$\frac{z'}{z} = \frac{X_{\mathcal{L}}}{R} = k$$

which in the general form is $y = ax$, or a straight line making an angle θ with the reference; this set of parametric equations represents all possible loci. When the parameter (k) is a constant, the locus is the straight line shown in Fig. 6–8*c*.

Case III (Fig. 6–8*d*). A constant impedance and variable power factor. Determine the locus diagram for the impedance value.

$$\bar{Z} = R + jX$$

$$Z = a$$

$$z = a\cos\theta$$

$$z' = a\sin\theta$$

are the parametric equations for a circle with the center at the origin (representing all possible loci as the parameter θ is changed and a is constant). If a is a constant, the form of the locus becomes $R^2 + X^2 = Z^2$, the locus given by the figure for all values of θ from 0 to 90°.

Case IV (Fig. 7–8). Constant voltage, constant resistance with variable inductance and current. Determine the current locus.

$$\bar{V} = \bar{I}(R + jX_{\mathcal{L}})$$
$$v + jv' = (i + ji')(R + jX_{\mathcal{L}})$$
$$v + jv' = iR - i'X_{\mathcal{L}} + j(i'R + iX_{\mathcal{L}})$$
$$v = iR - i'X_{\mathcal{L}}$$
$$v' = i'R + iX_{\mathcal{L}}$$

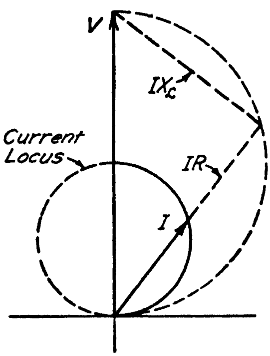


FIG. 7–8. Locus diagram for constant voltage and resistance with variable inductance and current.

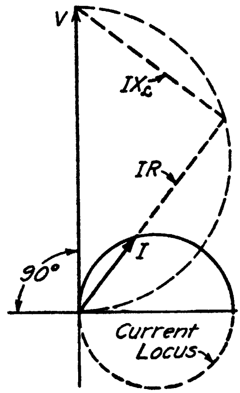


FIG. 8–8. Locus for constant voltage and inductance with variable resistance and current.

To eliminate $X_{\mathcal{L}}$ from this set of parametric equations, multiply v by i and v' by i' and add the equations

$$iv = i^2R - ii'X_{\mathcal{L}}$$
$$i'v' = i'^2R + ii'X_{\mathcal{L}}$$
$$(i^2 + i'^2)R - vi - v'i' = 0$$

which are in the form $Ax^2 + Ay^2 - Bx - Cy = 0$. This is a circle having a radius $v/2R$ with the center at $v/2R$, $v'/2R$, which point lies on the voltage vector $\bar{V}$. No physical significance can be placed on a leading current in this locus diagram.

Case V (Fig. 8–8). Constant voltage, constant inductance with variable resistance and current. Determine the current locus.

$$\bar{V} = \bar{I}(R + jX_{\mathfrak{L}})$$

$$v = iR - i'X_{\mathfrak{L}}$$

$$v' = i'R + iX_{\mathfrak{L}}$$

To eliminate R, multiply v by i' and v' by i and subtract one equation from the other:

$$(i^2 + i'^2)X_{\mathfrak{L}} - v'i + vi' = 0$$

which is the form $Ax^2 + Ay^2 - Bx + Cy = 0$. This is a circle with a radius $v/2X_{\mathfrak{L}}$, with a center at $v'/2X_{\mathfrak{L}}$, $-v/2X_{\mathfrak{L}}$, which point lies on a line at quadrature to the voltage vector $\bar{V}$. In the circuit under investigation, the current cannot lead the voltage nor can it lag the voltage by more than 90°; therefore, that portion of the locus not within this region has no physical significance.

This last case represents the condition encountered in the rotor of the induction motor under normal load conditions, where the change in reactance is neglected and the shaft load represents a variable resistance. The circle diagrams (loci diagrams) of inductance and resistance as influencing the current under constant voltage conditions may be widely applied in machines and circuits.

PROBLEMS

1–8. A voltage, $\bar{V} = 60 + j80$, is impressed upon an impedance consisting of a resistance of 8 ohms and an inductive reactance of 6 ohms. What is the voltage across the resistance and across the reactance? Give both the complex and absolute value. Draw the vector diagram for the circuit. How much current will flow if the same effective d-c voltage is applied?

2–8. A series circuit consisting of a resistance of 5 ohms and a capacitive reactance of 10 ohms is connected to a voltage, $\bar{V} = 150/\underline{120°}$. Determine the current and the voltage across each parameter. Draw the vector diagram. How much current will flow if the same effective d-c voltage is applied?

3–8. A resistance and a capacitance of 332 microfarads are connected in series across a 60-cycle a-c voltage, $\bar{V} = 200$ *cjs* 90°. If a current of 20 amp flows in the circuit determine (*a*) the resistance of the circuit, (*b*) the complex expression for the current, (*c*) the voltage across each part of the circuit. Draw the vector diagram.

4–8. A series circuit consists of a resistance of 16 ohms, an inductive reactance of 20 ohms, and a capacitive reactance of 8 ohms, and has a current flow of $\bar{I} = 10$ *cjs* 135°. Determine the voltage across each component part and the total voltage in absolute and complex form. Draw a vector diagram.

5–8. The following information is taken from a vector diagram for a series circuit: $\bar{I} = 16 + j12$, $\bar{V}_1 = 25 + j25$, $\bar{V}_2 = 50 - j15$, $\bar{V}_3 = 25 - j10$, and $\bar{V}_T = 100 + j0$. Determine the values of the impedances in the circuit. If the frequency is 50 cycles

what will be the values of (*a*) the resistance, (*b*) the inductance, and (*c*) the capacity in each impedance.

6–8. A 100-volt a-c constant voltage source is connected to a series circuit consisting of a resistance and a reactance. At a frequency of 25 cycles the current is 14.14 amp, and when the frequency is increased to 43.3 cycles the current decreases to 10 amp. Determine the character of the impedance.

7–8. An impedance is connected to a variable frequency source. At a 60-cycle frequency $\bar{V} = 100\underline{/90°}$ and $\bar{I} = 10\underline{/30°}$. If the voltage is 100 volts at a frequency of 25 cycles compare the currents and power factors at the two frequencies.

8–8. A series system containing resistance, inductance, and capacitance is connected to a 25-cycle source. The resistance is 8.66 ohms, the power factor is 92.2 per cent leading, and the current is 10 amp. If the frequency is increased to 43.3 cycles with the same voltage value applied, the current is 10 amp and the power factor is 92.2 per cent lagging. What is the value of the inductance and the capacity?

9–8. An impedance, $\bar{Z} = 40 + j30$, is connected to a 100-volt, 60-cycle source. How much capacitance must be connected in series with this impedance to cause a current of 2.5 amp to flow?

10–8. A circuit consisting of $\bar{Z}_1 = 80 + j60$, $\bar{Z}_2 = 40 - j40$, and $\bar{Z}_3 = 20 + j0$ in series is connected to a 100-volt, 60-cycle source. To what frequency should the source be changed for series resonance? If the applied voltage at resonance is 100 volts what is the voltage across each impedance?

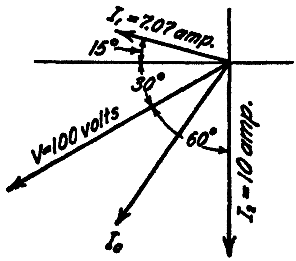
11–8. A current, $\bar{I} = 10 \; cjs \; 30°$, flows in an impedance $\bar{Z} = 10 \; cjs \; -60°$. Determine the complex expression for the voltage across the impedance and each component of the impedance.

12–8. A current, $\bar{I} = 5 + j8.66$, flows in a circuit consisting of three impedances in series connected to $\bar{V} = 50 + j86.6$. Two of the impedances are $\bar{Z}_1 = 2 + j2$ and $\bar{Z}_2 = 4 + j8$. Determine the voltage (complex and absolute values) across each impedance.

13–8. The voltage impressed upon an impedance is $\bar{V} = 80 - j60$ and is constant for all frequencies. At 50 cycles the impedance is $6 + j10$. If the system frequency is decreased to 30 cycles compare (*a*) current, (*b*) power factor, (*c*) voltage across each component of the impedance.

14–8. A current of 10 amp flows in a circuit consisting of $\bar{Z}_1 = 4 + j10$ and $\bar{Z}_2 = 4 - j4$ connected in series. The frequency is 60 cycles. If the frequency is decreased to 45 cycles determine the voltage required to maintain the current the same. Has the voltage been increased or decreased and how much?

15–8. Draw the electrical circuit which satisfies the vector diagram shown in the accompanying figure. Give all impedances in complex form.



PROB. 15–8.

CHAPTER 9

CIRCUIT PARAMETERS IN PARALLEL

Parallel circuits are the circuits commonly dealt with by the power engineer because practically all power transmission systems are of the parallel type. The problem is one of vector addition of currents with the voltage as a reference. Ohm's Law of any system is

$$V_m = I_m Z$$

and, if each side is divided by $\sqrt{2}$, it becomes

$$V = IZ$$

This form is satisfactory in dealing with the series combinations, but for parallel combinations the form

$$I = \frac{V}{Z} = VY$$

is used.

It may be observed that the impedance Z and the admittance Y are proportionality factors between voltages and currents:

$$Z = \frac{V}{I}$$

$$Y = \frac{I}{V} = \frac{1}{Z}$$

When expressed in symbolic form, the impedance and the admittance take the form of complex operators:

$$\bar{V} = \bar{I}\bar{Z} = \bar{I}(R + jX)$$

$$\bar{I} = \bar{V}\bar{Y} = \bar{V}(g + jb)$$

In the solution of parallel branches, if only one generating source is involved, it is better to depend upon the admittance Y than on the impedance. In the series system, impedance Z was used because the problem was one of voltage addition. In the parallel system, the problem is one of current addition. This has been demonstrated in Chapter 6, which deals with the general circuit.

1. Relationship between Impedance and Admittance. It has already been shown that $\bar{Z} = R \pm jX$ and, since Y is a reciprocal of Z, Y may be written

$$\bar{Y} = \frac{1}{\bar{Z}} = \frac{1}{R \pm jX}$$

If this latter expression is evaluated, then

$$\bar{Y} = \frac{1}{R \pm jX} \times \frac{R \mp jX}{R \mp jX}$$

$$= \frac{R}{(R^2 + X^2)} + j\frac{\mp(X)}{(R^2 + X^2)}$$

$$= g \mp jb \tag{a-9}$$

where (as previously shown)

$$X = (X_{\mathcal{L}} - X_C)$$

Therefore,

$$+j\frac{-(X)}{(R^2 + X^2)} = +j\frac{-(X_{\mathcal{L}} - X_C)}{(R^2 + X^2)} = +j\left[\frac{X_C}{(R^2 + X^2)} - \frac{X_{\mathcal{L}}}{(R^2 + X^2)}\right]$$

$$= +j(b_C - b_{\mathcal{L}})$$

Substituting this in equation (a–9),

$$\bar{Y} = g \mp jb = g + j(b_C - b_{\mathcal{L}})$$

In the expression for admittance the imaginary component (susceptance) has a positive or negative sign, depending upon whether the capacity or inductance predominates. The reverse is true for the expression for impedance. The above development fixes the relationships between the susceptance, conductance, reactance, and resistance of an electrical system.

$$g = \frac{R}{(R^2 + X^2)} = \frac{R}{Z^2}$$

$$b = \frac{X}{(R^2 + X^2)} = \frac{X}{Z^2} \quad \text{(sign by inspection)}$$

where

$$X = (X_{\mathcal{L}} - X_C)$$

$$b = (b_C - b_{\mathcal{L}})$$

Expressing the resistance and reactance in terms of conductance and susceptance,

$$\bar{Z} = \frac{1}{g \mp jb}$$

$$\bar{Z} = \frac{g}{g^2 + b^2} \pm j\frac{b}{g^2 + b^2}$$

$$R = \frac{g}{Y^2}$$

$$X = \frac{b}{Y^2} \quad \text{(sign by inspection)}$$

When dealing with the symbolic form of susceptance and the reactance, it is necessary to observe the fact that capacitance gives a negative sign to reactance and a positive sign to susceptance, whereas inductance gives a positive sign to reactance and a negative sign to susceptance.

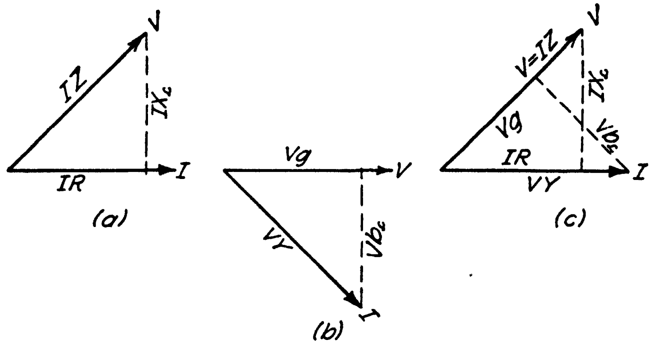


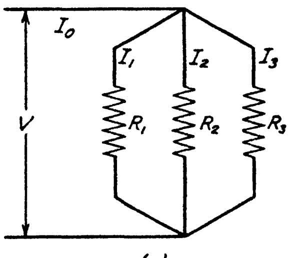
FIG. 1–9. Diagrams showing the relationships between resistance, reactance, impedance, conductance, susceptance, and admittance.

These signs must be observed in the complex operator. Figure 1–9 shows the relationship between the various currents and voltages when each system of notation assumes a different reference (pointed out in Chapter 6). It will be observed that the diagrams are drawn for voltages and currents and are not diagrams of circuit parameters. Circuit parameters are not vectors but linear quantities and are measured in units of ohms and mhos. However, if the scale of the diagram is changed in proportion to the values of V and I, the remaining right triangles, which are called the impedance and admittance triangles, will give the relationship of the parameter values. It should be observed that, in

dealing with impedances and admittances, the complex quantities of these values are added as well as the complex quantities representing vector voltages and currents; however, with impedances and admittances it does not imply the addition of vectors. In Fig. 1–9, the voltage triangle and the current triangle have, in each instance, a common factor of voltage and current, respectively. If each side of the triangle in Fig. 1–9*a* is divided by I, the resultant triangle is an impedance triangle, where the value Z may be expressed by a complex quantity. It is not, however, a vector diagram since the sides have only magnitude. By the same kind of analysis, an admittance triangle can be obtained by removing the common factor V from Fig. 1–9*b* (triangle with sides g, b, and Y.)

2. Treatment of Parallel Branches. Kirchhoff's Law, as stated for instantaneous values, may be expressed to include vector values as follows: The vector sum of the currents into or out of a junction is zero. The law then becomes all inclusive, covering cases of constant current as well as sinusoidal currents and instantaneous values. The application of this law to parallel branches establishes a means of developing branch combinations and will give the parameters for equivalent circuits needed in solving problems.

3. Parallel Resistances. Figure 2–9 shows three resistances in parallel with a voltage V across the terminals. The voltage may be for either



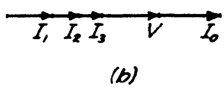


FIG. 2–9. A parallel circuit containing three resistances, and the vector diagram for such a system when a voltage is applied.

alternating current or direct current. Since the resistances are in parallel, the voltages across the various branches are the same and the current flowing in each branch, caused by the voltage, is controlled by the opposition in that individual branch. The total current, however, will be the vector sum of the branch currents.

The expression for current will be

$$I_0 = I_1 + I_2 + I_3$$

but the value for the current in any branch is

$$\bar{I}_1 = \frac{\bar{V}}{\bar{Z}_1} \qquad \bar{I}_2 = \frac{\bar{V}}{\bar{Z}_2} \qquad \bar{I}_3 = \frac{\bar{V}}{\bar{Z}_3}$$

Since the reciprocal of the impedance is the admittance,

$$\bar{I}_1 = \bar{V}\bar{Y}_1 \qquad \bar{I}_2 = \bar{V}\bar{Y}_2 \qquad \bar{I}_3 = \bar{V}\bar{Y}_3$$

and

$$\bar{I}_0 = \bar{V}(\bar{Y}_1 + \bar{Y}_2 + \bar{Y}_3) = \bar{V}\bar{Y}_0$$

or, in a parallel system of resistance branches, the total equivalent opposition is the reciprocal of the sum of the complex admittances. Evaluation of the admittances gives

$$\begin{aligned} \bar{Y}_0 &= (\bar{Y}_1 + \bar{Y}_2 + \bar{Y}_3) \\ &= (g_1 + g_2 + g_3) + j(\pm b_1 \pm b_2 \pm b_3) \end{aligned}$$

but, since the susceptance is zero (there being only resistances in this system), the total admittance is the algebraic sum of the conductances.

The expression for conductance is given by

$$g = \frac{R}{R^2 + X^2} = \frac{R}{Z^2}$$

but, since reactance is not present in a pure resistance system,

$$g = \frac{R}{R^2 + X^2} = \frac{R}{R^2} = \frac{1}{R}$$

and

$$g_1 = \frac{1}{R_1} \qquad g_2 = \frac{1}{R_2} \qquad g_3 = \frac{1}{R_3}$$

This will be true whether the system is operating on alternating or direct current. The total opposition in the system will be the reciprocal of the algebraic sum of the reciprocals of the resistances.

Figure 2–9*b* shows the vector relationship between the voltage and the currents in the system when alternating current is applied. In the d-c system, the values of current obtained in each branch are scalar values and would, therefore, be added by algebraic summation.

Since two and three branch combinations are often found in practice, the expressions given below are very useful.

Two resistances in parallel:

$$\frac{1}{R_0} = \frac{1}{R_1} + \frac{1}{R_2}$$

and the equivalent resistance

$$R_0 = \frac{R_1R_2}{R_1 + R_2}$$

Three resistances in parallel:

$$\frac{1}{R_0} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

and the equivalent resistance

$$R_0 = \frac{R_1R_2R_3}{R_1R_2 + R_1R_3 + R_2R_3}$$

Example a. Determine the equivalent resistance, the current in each branch, and the total current for both alternating and direct current in a system of three resistances (1, 2, and 3 ohms, respectively) in parallel across a 100-volt system.

$$\begin{aligned} R_0 &= \frac{R_1R_2R_3}{R_1R_2 + R_1R_3 + R_2R_3} \\ &= \frac{1 \times 2 \times 3}{(1 \times 2) + (1 \times 3) + (2 \times 3)} = \frac{6}{11} \\ &= 0.545 \text{ ohm} \\ I_1 &= VY_1 = 100 \times \tfrac{1}{1} = 100 \text{ amp} \\ I_2 &= VY_2 = 100 \times \tfrac{1}{2} = 50 \text{ amp} \\ I_3 &= VY_3 = 100 \times \tfrac{1}{3} = 33.3 \text{ amp} \\ I_0 &= I_1 + I_2 + I_3 = 183.3 \text{ amp} \end{aligned}$$

Check:

$$I_0 = \frac{V}{R_0} = \frac{100}{0.545} = 183.3 \text{ amp}$$

If 100 volts of direct current were applied to the above system, the flow of current would still be 183.3 amp. In this problem, the alternating current and voltage would be in phase. In a d-c system the voltage and current are always in phase.

In the a-c system, the term admittance is used. In the d-c system, only the first part of this term, the conductance, represents the same characteristic of the system, for in the steady-state condition of the d-c system the reactance is zero.

4. Reactances in Parallel. Since inductance and capacitance give reactances that are relatively opposite, any system containing either or both will be governed by the characteristic of the one that predominates. Figure 3–9 shows three reactances in parallel, two of which are pure capacitance and inductance, and the third is a combination of the two.

The first equations in Art. 3 are applicable to reactance as well as to resistance.

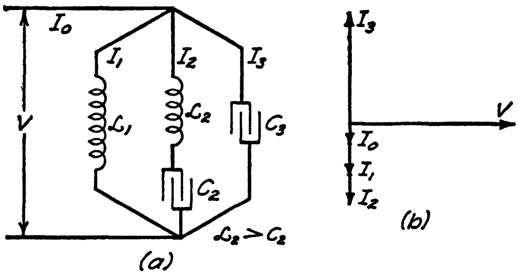


FIG. 3–9. Circuit containing three reactance branches in parallel, and the vector diagram when sinusoidal voltage is applied to the system.

This system (Fig. 3–9) being free from resistance, the conductance (g) is equal to zero. There remains only the susceptance to control the flow of current. The expression for the admittance is

$$\overline{Y}_0 = j(-b_1 - b_2 + b_3)$$

where

$$b = \frac{X}{R^2 + X^2} \quad \text{(sign by inspection)}$$

Since R is not present in the branch, the expression is

$$b = \frac{1}{X}$$

$$b_1 = -\frac{1}{X_1} \qquad b_2 = -\frac{1}{X_2} \quad \text{(see Fig. 3–9)} \qquad b_3 = \frac{1}{X_3}$$

The sign will depend upon whether the capacitance or the inductance predominates. With an excess of capacitance the sign will be positive; with an excess of inductance the sign will be negative. It has been shown, in Chapter 4, that there will be no opposition to the flow of current in a d-c system containing pure inductive reactance; therefore, the current will be infinite for any finite voltage. It is to be understood that pure capacitance and inductance are impossible, and the treatment given here applies only to hypothetical examples. It does not demon-

strate practical conditions, but it does establish a familiarity with the component parts of circuit phenomena so that the actual combinations found will be more clearly understood. Figure 3-9*b* shows the vector relationships existing when alternating current is applied to the circuit.

Example b. Determine the equivalent susceptance of a circuit of three parallel branches where one branch has an inductance of 0.01 henry, the second contains an inductance of 0.01 henry and a capacity of 2650 microfarads in series, and the third has a capacity of 530 microfarads. When a 60-cycle voltage of 100 volts is applied to the system, what will be the total current and how much current will flow in each branch?

$$X_1 = 2\pi f\mathcal{L} = 2 \times 3.1416 \times 60 \times 0.01 = 3.77 \text{ ohms}$$

$$X_2 = \left(2\pi f\mathcal{L} - \frac{10^6}{2\pi fC}\right) = (3.77 - 1) = 2.77 \text{ ohms}$$

$$X_3 = \frac{1}{2\pi fC} = \frac{10^6}{2 \times 3.1416 \times 60 \times 530} = 5.0 \text{ ohms}$$

$$b_1 = \frac{1}{3.77} = 0.266 \quad \text{(with negative sign)}$$

$$b_2 = \frac{1}{2.77} = 0.361 \quad \text{(with negative sign)}$$

$$b_3 = \tfrac{1}{5} = 0.2$$

$$b_0 = b_1 + b_2 + b_3$$

$$= 0.427 \quad \text{(with a negative sign)}$$

or, by complex notation,

$$\bar{Y}_0 = \bar{Y}_1 + \bar{Y}_2 + \bar{Y}_3$$

$$= (0 - j0.266) + (0 - j0.361) + 0 + j0.2) = 0 - j0.427$$

The equivalent susceptance will be 0.427 mho and the equivalent reactance is inductive.

$$\bar{I}_1 = (100 + j0)(0 - j0.266) = |\,26.6\,| \text{ amp}$$

$$\bar{I}_2 = (100 + j0)(0 - j0.361) = |\,36.1\,| \text{ amp}$$

$$\bar{I}_3 = (100 + j0)(0 + j0.2) = |\,20\,| \text{ amp}$$

in which both I_1 and I_2 lag the voltage, whereas I_3 leads, the power factor angle in each instance being 90°.

The total current will be the vector sum of the individual branches

$$\bar{I}_0 = \bar{I}_1 + \bar{I}_2 + \bar{I}_3 = 0 - j42.7$$

$$I_0 = 42.7 \text{ amp lagging the voltage by } 90°$$

5. Equivalent Inductance and Capacitance of Parallel Inductances and Capacitances. Since the inductive reactance is $2\pi f\mathfrak{L}$ and the capacitive reactance is $1/2\pi fC$,

$$\frac{1}{X_0} = Y_0 = \frac{1}{X_1} + \frac{1}{X_2} + \frac{1}{X_3} \quad \text{(no } R \text{ in circuit branch)}$$

$$\frac{1}{2\pi f\mathfrak{L}_0} = \frac{1}{2\pi f\mathfrak{L}_1} + \frac{1}{2\pi f\mathfrak{L}_2} + \frac{1}{2\pi f\mathfrak{L}_3}$$

or

$$\frac{1}{\mathfrak{L}_0} = \frac{1}{\mathfrak{L}_1} + \frac{1}{\mathfrak{L}_2} + \frac{1}{\mathfrak{L}_3}$$

The inductances in parallel obey the same law as resistances in parallel· Combinations of two or three inductances in parallel will be

$$\mathfrak{L}_0 = \frac{\mathfrak{L}_1\mathfrak{L}_2}{\mathfrak{L}_1 + \mathfrak{L}_2}$$

$$\mathfrak{L}_0 = \frac{\mathfrak{L}_1\mathfrak{L}_2\mathfrak{L}_3}{\mathfrak{L}_1\mathfrak{L}_2 + \mathfrak{L}_1\mathfrak{L}_3 + \mathfrak{L}_2\mathfrak{L}_3} \quad \text{respectively}$$

For capacities in parallel the relationships are

$$Y_0 = \frac{1}{X_1} + \frac{1}{X_2} + \frac{1}{X_3}$$

$$\frac{1}{X_0} = 2\pi f C_0 = 2\pi f C_1 + 2\pi f C_2 + 2\pi f C_3$$

or

$$C_0 = C_1 + C_2 + C_3$$

Capacities in parallel are added to give the equivalent capacity of the system.

Table I–9 gives a summary of the treatment of resistances, inductances, and capacitances in parallel and series. The frequency with which resistance appears in normal calculations and the frequent neglect of reactance leads to a familiarity with resistance conditions that is lacking in inductance and capacitance. The table assumes that all other parameters are to be neglected when the combinations shown are considered.

6. Parallel Branches Which Have Combinations of Parameters. Since all circuits contain resistance, inductance, and capacitance to some degree, the normal system is a combination of resistance and reactance.

TABLE I-9

Parameter	Series Connection	Parallel Connection
Pure resistance (no $\mathcal{L}$ and C)	$R_0 = R_1 + R_2 + R_3$	$\frac{1}{R_0} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$
Pure inductance (no R and C)	$\mathcal{L}_0 = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$	$\frac{1}{\mathcal{L}_0} = \frac{1}{\mathcal{L}_1} + \frac{1}{\mathcal{L}_2} + \frac{1}{\mathcal{L}_3}$
Pure capacitance (no R and $\mathcal{L}$)	$\frac{1}{C_0} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$	$C_0 = C_1 + C_2 + C_3$
Pure inductive reactance (no R and C)	$X_0 = X_{\mathcal{L}_1} + X_{\mathcal{L}_2} + X_{\mathcal{L}_3}$	$Y_0 = \frac{1}{X_{\mathcal{L}_1}} + \frac{1}{X_{\mathcal{L}_2}} + \frac{1}{X_{\mathcal{L}_3}}$
Pure capacitive reactance (no R and $\mathcal{L}$)	$X_0 = X_{C_1} + X_{C_2} + X_{C_3}$	$Y_0 = \frac{1}{X_{C_1}} + \frac{1}{X_{C_2}} + \frac{1}{X_{C_3}}$

In transmission network calculations much of the work is based on lines with negligible resistance, whereas in secondary distribution it is not uncommon to neglect the reactance if the wire is not too large and the spacing is equal to that in normal building wiring; this should give a small value of reactance.

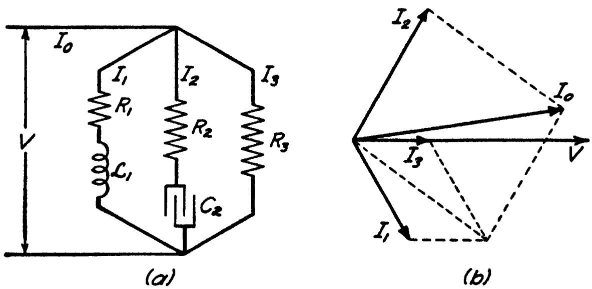


FIG. 4-9. Circuit containing three impedance branches in parallel, and the vector diagram when a sinusoidal voltage is applied.

Figure 4-9 shows three impedances in parallel: the first composed of inductance and resistance; the second, of capacitance and resistance; the third, a pure resistance. The solution of this problem is similar to the two hypothetical examples mentioned before; however, the admittance will contain both conductance and susceptance:

$$\overline{Y}_0 = (g_1 + g_2 + g_3) + j(-b_1 + b_2 + b_3)$$

where

$$g = \frac{R}{R^2 + X^2} = \frac{R}{Z^2}$$

$$b = \frac{X}{R^2 + X^2} = \frac{X}{Z^2} \quad \text{(sign by inspection)}$$

After determining the values for the various susceptances and conductances in the branches, the sign of the susceptance should be checked by physical investigation of the parameters of the branch. The conductance and susceptance will determine the admittance and this, together with the voltage, determines the current flow. This problem is more easily solved by using symbolic notation and obtaining the complex expression for the equivalent impedance.

To determine the equivalent impedance of the system (Fig. 4–9) it is first necessary to add the admittances determined from the individual impedances:

$$\bar{Z}_1 = R_1 + j2\pi f \mathcal{L}_1 = R_1 + jX_1$$

$$\bar{Z}_2 = R_2 - j\frac{1}{2\pi f C_2} = R_2 - jX_2$$

$$\bar{Z}_3 = R_3 + j0$$

which can be combined to give

$$\frac{1}{\bar{Z}_0} = \frac{1}{\bar{Z}_1} + \frac{1}{\bar{Z}_2} + \frac{1}{\bar{Z}_3}$$

or

$$\bar{Z}_0 = \frac{\bar{Z}_1\bar{Z}_2\bar{Z}_3}{\bar{Z}_1\bar{Z}_2 + \bar{Z}_2\bar{Z}_3 + \bar{Z}_1\bar{Z}_3}$$

Often it will be found more convenient to evaluate $1/\bar{Z}_1$, $1/\bar{Z}_2$, and $1/\bar{Z}_3$, add them, and take the reciprocal of the sum of the terms in determining equivalent impedance of the system. Only developed judgment will lead to the proper choice of method.

Given the value of the equivalent impedance, the total current is

$$\bar{I}_0 = \frac{\bar{V}}{\bar{Z}_0}$$

The other functions of the circuit may be determined from these relationships. The value of the equivalent impedance in terms of resist-

ance and reactance is

$$Z_0 = \left[\frac{(R_1 + jX_1)(R_2 + jX_2)(R_3 + jX_3)}{(R_1+jX_1)(R_2+jX_2) + (R_1+jX_1)(R_3+jX_3) + (R_2+jX_2)(R_3+jX_3)}\right]$$

$$Z_0 = \left[\frac{\begin{Bmatrix} R_1R_2R_3 - R_1X_2X_3 - R_2X_1X_3 - R_3X_1X_2 \\ + j(R_1R_3X_2 + R_1R_2X_3 + R_2R_3X_1 - X_1X_2X_3) \end{Bmatrix}}{\begin{Bmatrix} (R_1R_2 + R_1R_3 + R_2R_3) - (X_1X_2 + X_1X_3 + X_2X_3) \\ + j[X_1(R_2 + R_3) + X_2(R_1 + R_3) + X_3(R_1 + R_2)] \end{Bmatrix}}\right]$$

where $X = X_L - X_C$. By using symbolic notation, it is possible to develop these general expressions and use them in making system studies. When specific individual terms vary, the remaining terms may be grouped under a constant. Development of the general form is tedious, but when many calculations of the same nature must be made, general developments save time.

The equivalent impedance determined in this general example applies directly to systems using alternating current. When the same impedance system is to be studied in d-c applications, limiting conditions exist in each branch. To determine the equivalent opposition to direct current in a system, remove all the branches containing series capacitance and from the remaining branches remove all terms representing inductance. When the above general impedance development is applied to the circuit in Fig. 4–9, R_1 and R_3 are the only oppositions operative since R_2 now ceases to be a branch of the system because there is capacitance in series and the system reduces to two parallel branches. The expression for the total impedance takes the form

$$Z_1 = (R_1 + jX_1) = R_1 + j0$$

$$Z_2 = \infty \quad \text{(represents infinite opposition, therefore an open branch)}$$

and

$$Z_3 = R_3 + jX_3 = R_3 + j0$$

in which the inductances are not operative and the opposition of the two finite impedances is reduced to the resistances R_1 and R_3, making the total impedance, when opposing d-c flow,

$$Z_0 = R_0 + j0 \qquad R_0 = \frac{R_1R_3}{R_1 + R_3}$$

The system becomes two parallel branches across a common voltage. Whenever a circuit carries both direct and alternating current, as fre-

quently occurs in tube and signal circuits, the individual conditions must be studied and each part analyzed separately.

Example c. Given three impedances (Fig. 5–9) in parallel, determine the total current and the power factor if a voltage of 100 volts, 60 cycles, is applied to the terminals. Draw the vector diagram. How much current would flow if 100 volts direct current were applied to the terminals?

$$\frac{1}{\bar{Z}_0} = \frac{1}{1+j1} + \frac{1}{1-j1} + \frac{1}{2+j0}$$

$$= \frac{2(1+j1) + 2(1-j1) + (1+j1)(1-j1)}{2(1+j1)(1-j1)}$$

$$\bar{Y}_0 = \frac{1}{\bar{Z}_0} = \frac{2+j2+2-j2+2}{4+j0} = \frac{6+j0}{4+j0} = 1.5+j0$$

$$\bar{Z}_0 = \frac{1}{1.5+j0} = 0.667+j0$$

$$\bar{I}_0 = \bar{V}_0\bar{Y}_0 = (100+j0)(1.5+j0) = 150+j0$$

$$I_0 = 150 \text{ amp}$$

$$\text{Power factor: } \cos\theta = 1 \qquad \theta = 0°$$

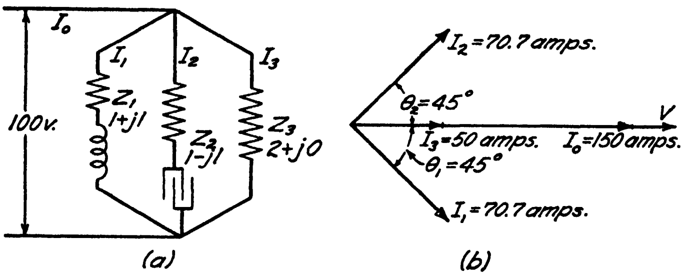


FIG. 5–9. Circuit and vector diagrams for a system of three impedances in parallel placed across a sinusoidal voltage.

For direct current, the middle branch which contains a condenser would disappear and the expression for the two remaining branches would be

$$R_0 + jX_0 = \bar{Z}_0 = \frac{(R_1+jX_1)(R_3+jX_3)}{(R_1+R_3)+j(X_1+X_3)}$$

Neglecting X_L,

$$R_0 = \frac{R_1R_2}{R_1+R_2} \quad \text{(or two resistances in parallel)}$$

$$R_0 = \frac{1\times 2}{1+2} = \frac{2}{3} = 0.667$$

$$I_0 = \frac{100}{0.667} = 150 \text{ amp}$$

The vector diagram is shown in Fig. 5–9*b*.

In this problem, the current for both direct and alternating current is the same.

7. Parallel (Phase) Resonance. When the system is in phase resonance the power factor is unity. This is the condition desired in a-c distribution, since it is the most efficient. However, in most instances, economy prohibits the obtaining of unity power factor. In a system where induction motors are used, the introduction of a condenser to improve the system power factor is an economic method of approaching resonance.

Figure 6–9 shows the resultant values of the current when an inductance has been paralleled with a condenser. The parallel impedances

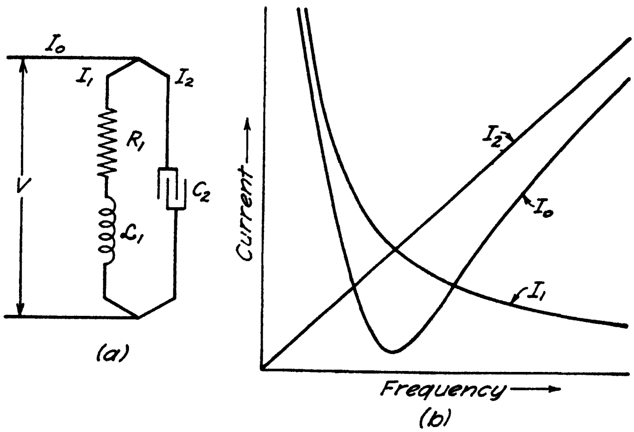


FIG. 6–9. Parallel resonance, showing the electrical circuit used and the curves for the currents as the frequency is changed.

have been placed across a constant voltage source and the frequency has been varied to determine the value of current flow at the various frequencies.

At resonance, the quadrature components of the current are equal and so phased that they cancel each other; therefore, the only effective opposition in the system is the resistance.

Parallel resonance (meaning parallel phase resonance) is the steady-state condition which exists in a circuit containing inductance and capacitance connected in parallel, when the current entering the circuit from the supply line is in phase with the voltage across the circuit.

In communication, either telephony or radio, the solution for resonant frequency is obtained by using the inductance and capacitance, but in power systems it is simpler to determine the expressions for the admittance of the various branches and the equivalent admittance of the circuit. The current values are obtained from these admittances and

the system voltage. For the two branch circuits shown, the general expressions for the current will be

$$\bar{I}_1 = \bar{V}_0 \frac{1}{R_1 + jX_1}$$

$$\bar{I}_2 = \bar{V}_0 \frac{1}{0 - jX_2}$$

$$\bar{I}_0 = \bar{V}_0 \left(\frac{1}{R_1 + jX_1} + \frac{1}{0 - jX_2} \right)$$

$$= \bar{V}_0 \left[\frac{R_1 + j(X_1 - X_2)}{X_1 X_2 - jR_1 X_2} \right] = \bar{V}_0 \bar{Y}_0$$

where $X = (X_L - X_C)$.

Summary—Parallel Circuits

(1) Currents add vectorially: $\bar{I}_0 = \bar{I}_1 + \bar{I}_2 + \bar{I}_3 \cdots + \bar{I}_N$

(2) Admittances add vectorially: $\bar{Y}_T = \bar{Y}_1 + \bar{Y}_2 + \bar{Y}_3 \cdots + \bar{Y}_N$

(3) Conductances add numerically: $g_T = g_1 + g_2 + g_3 \cdots + g_N$

(4) Susceptances add algebraically: $b_T = \pm b_1 \pm b_2 \pm b_3 \cdots \pm b_N$

(5) The voltage vector is generally chosen as the reference vector for drawing vector diagrams.

(6) $$Y = \sqrt{g^2 + b^2} \quad \text{or} \quad \bar{Y} = g \pm jb.$$

(7) Power factor: $\cos \theta = \dfrac{g}{Y}$

8. Series-Parallel Circuits. When impedances are connected in series and such groups are then paralleled, or when impedances are connected in parallel and such groups are then placed in series, there exists what is known as a series-parallel circuit. It is possible to obtain an equivalent series impedance which would draw the same amount of current and have the same power factor as the original combination. This equivalent system is obtained by a process of telescoping; therefore, the analysis of the equivalent impedance begins with the elements farthest from the source. To telescope a system it is necessary that there be no electromotive forces except a single supply. Whenever electromotive forces other than the supply voltage are impressed on the system, it must be treated as a network and the methods discussed in the next chapter must be applied.

Figure 7–9 shows a line feeding two impedance loads. Each branch contains an impedance; therefore, the combination forms a series-parallel system—the most common system in power distribution. To solve this system, the impedances Z_2 and Z_3, being in series, are combined:

$$\bar{Z}_{23} = \bar{Z}_2 + \bar{Z}_3$$

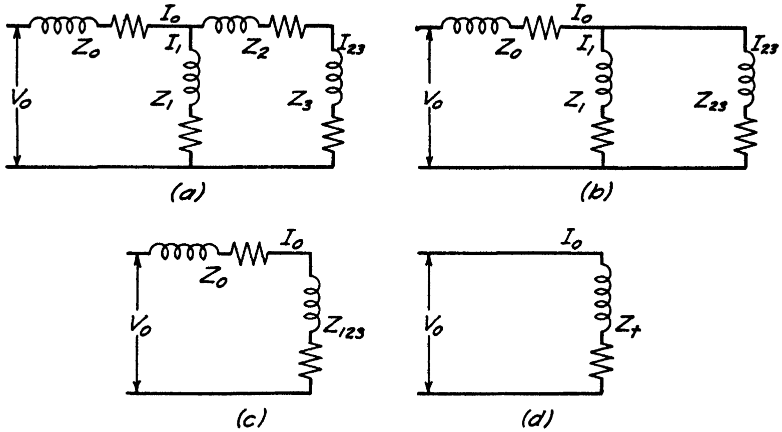


FIG. 7–9. Telescoping a series-parallel system to determine the equivalent impedance of the system.

Figure 7–9c shows the reduction of the system to an equivalent system, with Z_{23} replacing the two impedances Z_2 and Z_3. The impedances Z_{23} and Z_1 are in parallel and may be combined to form an equivalent impedance by the addition of admittances:

$$\bar{Y}_{123} = \frac{1}{\bar{Z}_1} + \frac{1}{\bar{Z}_{23}}$$

$$\bar{Z}_{123} = \frac{\bar{Z}_1\bar{Z}_{23}}{\bar{Z}_1 + \bar{Z}_{23}}$$

Figure 7–9c shows the replacement of $\bar{Z}_1$, $\bar{Z}_2$, and $\bar{Z}_3$ by an impedance $\bar{Z}_{123}$ in series with the impedance $\bar{Z}_0$ which, when added, is

$$\bar{Z}_t = \bar{Z}_0 + \bar{Z}_{123}$$

This is the equivalent impedance of the system; it is capable of reacting to the alternating current in the same manner as the original system, drawing the same current at the same power factor as this system. From this point on, the system may be treated similarly to either the

parallel or series system in determining the current flow and the vector diagram that represents the system.

9. Locus Diagrams. The general purpose of the locus diagram has been discussed in Chapter 8. A study of the parameter loci for the parallel system can be classified under limitations set by the following equations.

$$\overline{Y} = g - jb_{\mathcal{L}}$$

$$\tan\theta = -\frac{b_{\mathcal{L}}}{g}$$

If the study is confined to the inductive system (which was the procedure used with the series circuit), the values of g, b, and Y are (Fig. 8–9)

$$\overline{Y}_1 = g_1 - jb_1$$

$$\overline{Y}_2 = g_2 - jb_2$$

$$\overline{Y} = (g_1 + g_2) - j(b_1 + b_2)$$

$$g_1 = \frac{1}{R} \qquad b_1 = 0 \qquad g_2 = 0 \qquad b_2 = -\frac{1}{X_{\mathcal{L}_2}} = -\frac{1}{2\pi f\mathcal{L}_2}$$

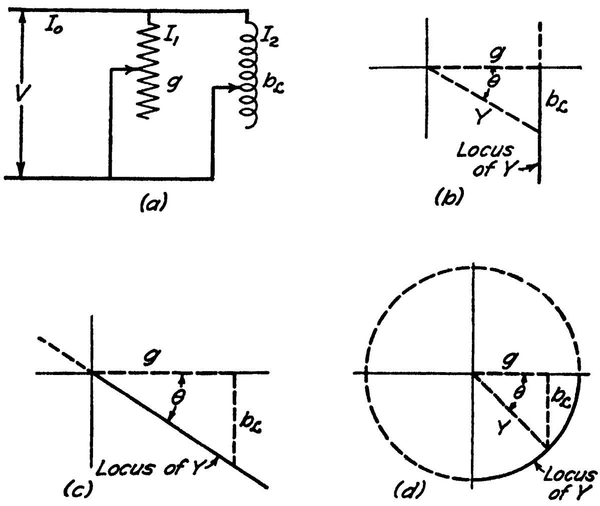


FIG. 8–9. Locus diagrams for admittance.

The first three cases in parallel parameters are analyzed in the same manner as in the series circuit:

Case I (Fig. 8–9*b*). The conductance (g) and the inductance ($\mathcal{L}$) are constant while the frequency is changed. The locus is a straight line.

Case II (Fig. 8–9*c*). A variable admittance with a constant power factor gives a straight line.

Case III (Fig. 8–9*d*). A constant scalar value of admittance with variable power factor has a circular locus.

A study of the voltage-current relationships with variable parallel parameters proves interesting because a major portion of the problems associated with machines and circuits can be classified under the headings of parallel or series-parallel systems.

Case IV (Fig. 9–9). Determine the locus of a system having a constant current, constant conductance with variable inductive susceptance,

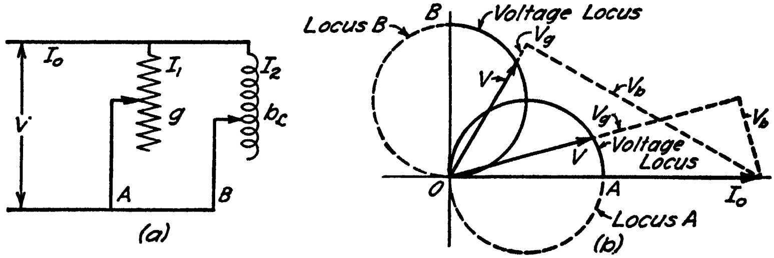


FIG. 9–9. Voltage locus diagrams in a system of constant current with (*A*) constant conductance and variable inductive susceptance and (*B*) variable conductance and constant inductive susceptance.

and variable voltage. Given $\bar{I} = \bar{V}(g - jb_x)$; by the same analysis as that used in series circuits, the resultant voltage locus will be a circle,

$$(v^2 + v'^2)g - vi + v'i' = 0$$

with a radius $\bar{I}/2g$, and center located at $i/2g$, $i'/2g$, which point lies on the current vector $\bar{I}$. The lagging voltage has no significance whenever capacitance is not present. The arc *OVA* in Fig. 9–9 represents the locus.

Case V (Fig. 9–9). Determine the locus of a system having a constant current, constant inductive susceptance with variable conductance, and variable voltage. The voltage locus will be a circle,

$$(v^2 + v'^2)b + vi' - v'i = 0$$

of radius $I/2b$, with its center located at $i/2b$, $-\ i'/2b$, which point lies on a line at 90° to the current vector $\bar{I}$. The arc *OVB*, in Fig. 9–9, represents the locus. The dotted portion of the locus diagram has no physical significance in this problem.

In addition to the five cases similar to those in the series-circuit discussion, there are three more conditions that show interesting loci. These are in a series-parallel system: (*a*) constant current from a con-

stant voltage source, (*b*) phase compensation using a parallel condenser, (*c*) the necessity of varying the sending end voltage in order to keep the receiving end voltage constant as the power factor of the constant load changes.

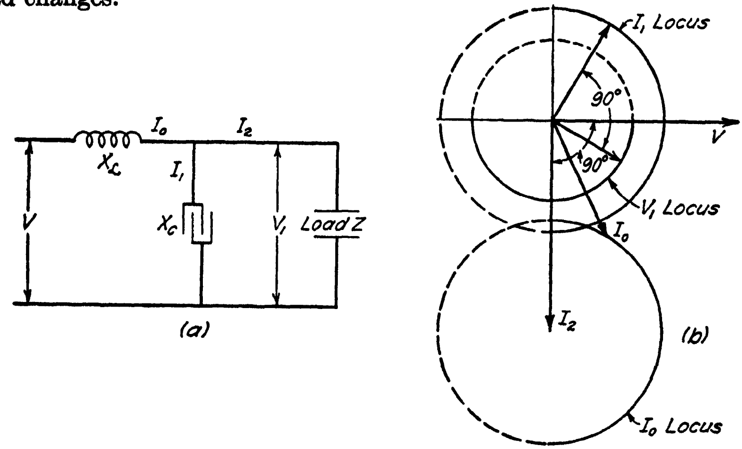


Fig. 10–9. Locus of current in a constant voltage to constant current conversion system (*b*), where Z is a scalar constant but a circular locus.

Case VI (Fig. 10–9). Determine the locus diagram of the current (I_2) in the load branch of the system if the load (Z) is varied while $X_{\mathcal{L}}$ and X_C are constant and equal and the voltage (V) remains constant.

$$\bar{I}_1 = \frac{\bar{V}_1}{-jX_C}$$

$$\bar{I}_2 = \bar{V}_1/\bar{Z}$$

$$\bar{I}_0 = \bar{I}_1 + \bar{I}_2 = \bar{V}_1\left(\frac{1}{\bar{Z}} - \frac{1}{jX_C}\right)$$

$$\bar{V} = \bar{V}_1 + j\bar{I}_0 X_{\mathcal{L}}$$

$$\bar{V} = \bar{V}_1 + jX_{\mathcal{L}}\bar{V}_1\left(\frac{1}{\bar{Z}} - \frac{1}{jX_C}\right)$$

$$\bar{V} = \bar{V}_1\left(1 + \frac{jX_{\mathcal{L}}}{\bar{Z}} - \frac{jX_C}{jX_C}\right)$$

since $X_{\mathcal{L}} = X_C$.

$$\bar{V} = \frac{\bar{V}_1 jX_{\mathcal{L}}}{\bar{Z}} = \bar{I}_2 jX_{\mathcal{L}}$$

$$V = I_2 X_{\mathcal{L}} \quad \text{or} \quad I_2 = \frac{V}{X_{\mathcal{L}}}$$

When V is constant, I_2 is constant and controlled in magnitude by $X_{\mathcal{L}}$ or

$$\bar{V} = \bar{I}_2(jX_{\mathcal{L}})$$

$$v + jv' = (i_2 + ji'_2)(jX_{\mathcal{L}})$$

$$v + jv' = ji_2X_{\mathcal{L}} - i'_2X_{\mathcal{L}}$$

$$v = -i'_2X_{\mathcal{L}}$$

$$v' = i_2X_{\mathcal{L}}$$

This is a set of parametric equations representing a series of loci of straight lines unchanging in length. Since $\bar{V}_1 = \bar{I}_2\bar{Z}$, the locus of V_1 will depend upon the locus of Z.

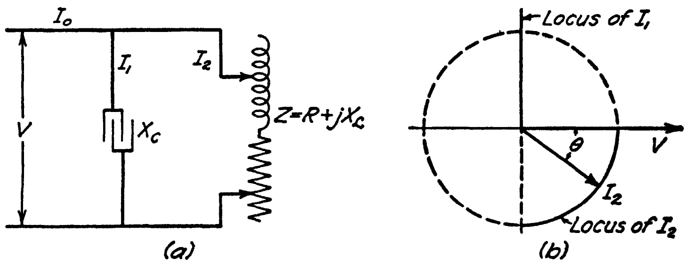


FIG. 11–9. Locus diagrams of load and condenser currents in system power-factor correction.

Case VII (Fig. 11–9). Determine the locus for the parallel condenser current if the load impedance is constant with a variable power factor, and the total power factor of the system is unity.

From Case III, the admittance locus will be a circle and, since $\bar{I}_2 = \bar{V}\bar{Y}$ with $\bar{V}$ a constant, the current locus is a circle and $\bar{I}_2 = i_2 - ji'_2$. For unity power factor, it is necessary that

$$\bar{I}_2 = i_2 - ji'_2$$

and

$$i'_2 = I_2 \sin\theta$$

$$i_2 = I_2 \cos\theta$$

These are the parametric equations for the current locus. It follows that I_1 is through the origin, at 90° leading, and equal in value at all times to $I_2 \sin\theta$.

Case VIII (Fig. 12–9). Find the locus of the sending end voltage which will maintain constant receiving end voltage when the power factor varies under constant volt-ampere load.

The value of V is expressed by

$$\bar{V} = \bar{V}_1 + j\bar{I}\bar{Z}$$

$$\bar{V} = \bar{V}_1 + \bar{I}(R + jX)$$

$$v + jv' = (v_1 + jv'_1) + (i + ji')(R + jX)$$

$$v + jv' = v_1 + jv'_1 + iR + jiX + ji'R - i'X$$

where

$$v = v_1 + iR - i'X$$

$$v' = v'_1 + iX + i'R$$

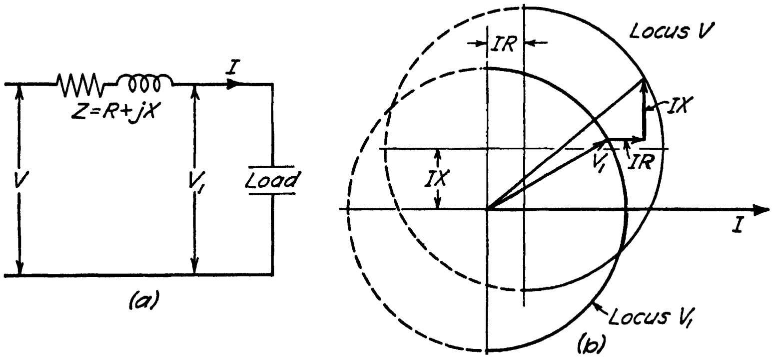


FIG. 12–9. Locus of receiving and sending end voltages for constant volt-ampere load and variable power factor.

These are the parametric equations for the sending end voltage and

$$v_1 = V_1 \cos \theta$$

$$v'_1 = V'_1 \sin \theta$$

are the parametric equations for the receiving end voltage. Since in the condition shown in Fig. 12–9, the current is the reference,

$$v = v_1 + IR$$

$$v' = v'_1 + IX$$

and the locus for both V and V_1 are as given.

The foregoing cases outline the method of approach and the analysis of some special problems by locus diagrams. This form of representation will be used in succeeding chapters which will discuss equivalent circuits of machines and distribution circuits.

PROBLEMS

1–9. Three resistances, $R_1 = 5$ ohms, $R_2 = 10$ ohms, and $R_3 = 2$ ohms, are connected in parallel across a 100-volt, 60-cycle system. (*a*) Determine the current flowing. (*b*) If the same effective d-c voltage is connected to the system what current will flow?

2–9. Two impedances, $\bar{Z}_1 = 4 + j3$ and $\bar{Z}_2 = 6 - j8$, are placed in parallel across $\bar{V}_0 = 80 + j60$. Solve for (*a*) total power, (*b*) power factor, and (*c*) complex value for the total current. (*d*) Draw a vector diagram for the a-c solution and determine (*e*) values of power and (*f*) current if the system is connected to the same effective d-c voltage.

3–9. (*a*) Construct the vector diagram for the parallel system given by the following data: $\bar{I}_0 = 150 + j0$, $\bar{V}_0 = 100 + j0$, $\bar{I}_1 = 50 - j50$, $\bar{I}_2 = 50 + j50$, $\bar{I}_3 = 50 + j0$. (*b*) Compute the values necessary to construct a circuit diagram giving values of voltage, current, and impedance in every part.

4–9. When a voltage, $\bar{V}_0 = 0 - j100$, causes 30 amp to flow through three impedances in parallel: $\bar{Z}_1 = 8 + j6$, $\bar{Z}_2 = 4 - j3$, and $\bar{Z}_3$, what is the value of $\bar{Z}_3$ which produces resonance?

5–9. A series-parallel system has two impedances, $\bar{Z}_3 = 1 + j2$ and $\bar{Z}_2 = 0 - j4$, in parallel and this combination is in series with $\bar{Z}_1 = 0 + j4$. If the supply is $\bar{V}_0 = 100 + j0$, determine (*a*) voltage and (*b*) current values in both complex and absolute form.

6–9. Two parallel impedances, $\bar{Z}_3 = 3 + j4$ and $\bar{Z}_4 = 4 - j3$, are supplied through two line impedances, Z_1 and Z_2, each equal to $1 + j1$. If a current, $I_4 = 10$ amp flows, what is the applied voltage?

7–9. Two impedances, $\bar{Z}_1 = 2 + j3$ and $\bar{Z}_2 = 3 - j2$, are connected in parallel across $\bar{V}_0 = 100 + j0$. Determine (*a*) all the currents, (*b*) voltages, (*c*) impedances, (*d*) power factors, and (*e*) power for the branches and line. With the same effective d-c voltage determine (*f*) the currents, (*g*) voltages, and (*h*) power as above.

8–9. Two parallel branches placed across a 100-volt a-c system require 10 amp with a 60 per cent lagging power factor. If the current in one branch is $\bar{I}_1 = 2 + j4$, what is the value of (*a*) $\bar{I}_0$, (*b*) $\bar{I}_2$, (*c*) the power in the line, and (*d*) the power in the branches; (*e*) the power factor of the branches?

9–9. A series impedance is expressed by $\bar{Z}_0 = 10 + j20$. (*a*) What is the equivalent parallel system? (*b*) Give the value of the complex admittance.

10–9. A two-parallel-branch system has an admittance of $\bar{Y}_o = 0.04 - j0.02$. Determine the complex expression for the impedance of the equivalent series system.

11–9. In a parallel system the equivalent impedance $\bar{Z}_0 = 2 - j1$. If the impedance of one branch is $\bar{Z}_1 = 1 - j3$, what is the impedance of the second branch?

12–9. What value of resistance in parallel with $\bar{Z}_1 = 1 + j1$ will give an equivalent impedance, Z_0 having a lagging power factor of 86.6 per cent?

13–9. Three impedances in parallel across $\bar{V}_0 = 86.6 - j50$. The currents $\bar{I}_1 = 5 + j8.66$, $\bar{I}_2 = 5 - j8.66$, and $\bar{I}_3 = 17.32 + j0$ flow in the parallel branches. Find the impedance for each branch.

14–9. In a system with three impedances in parallel and an impressed voltage, $\bar{V}_0 = -50 + j86.6$, the total current $I_0 = 20$ amp and leads the voltage by 30°, the current $I_1 = 10$ amp and leads the voltage by 60°, the current $I_2 = 10$ amp and is in phase with the voltage. Draw the circuit and mark on it the complex values of the impedances.

15–9. When the voltage, $\bar{V}_0 = 100 + j0$, is applied to a pure resistance $R_1 = 10$ ohms in parallel with a pure inductive reactance the line current has an 80 per cent lagging power factor. Find the value of (*a*) the inductive reactance and (*b*) all the currents in complex expressions.

CHAPTER 10

SOLUTION OF NETWORKS

Each operating electrical circuit contains a cause (emf), opposition (R, $\mathcal{L}$, and C), and effect (current), and this is true for each individual electrical circuit regardless of how many may be connected in series or in parallel in a complicated network. Kirchhoff's Laws state the relationships between these fundamental parts of the circuit during steady-state conditions, and a definite solution of any problem can be obtained as long as electrical equilibrium is maintained.

1. Kirchhoff's Laws. *Statement of the Laws.* Kirchhoff's Laws (previously stated in Chapter 6 for instantaneous values of currents and voltages) are also true when effective values of currents and voltages are used.

(*a*) *First Law: Current Relationship.* The vector sum of the currents flowing toward or away from any junction equals zero. $\Sigma\bar{I} = 0$. All currents cannot flow toward the junction at any instant since this would mean a piling up of current.

(*b*) *Second Law.* The vector sum of all voltages (rise or fall of potential) around a closed electrical circuit equals zero. $\Sigma\bar{V} = 0$. Any closed loop may contain both sources of electrical power (causes) and sinks (opposition) of electric power. The sum of the voltages around a closed circuit will include both source voltages and opposition voltages, or voltage drops across individual impedances.

Several interpretations can be placed upon the meaning of a source voltage and an opposition voltage. The effect of each in the electrical circuit is different and some designation is necessary to differentiate between them. One designation can be the use of the terms "source" and "sink"; another designation is the use of plus (+) and minus (−) signs. The different ways of stating Ohm's Law illustrate this point. Ohm's Law may be written in three ways:

(1) $$\bar{V} = \bar{I}\bar{Z}$$

(2) $$\bar{V} - (+\bar{I}\bar{Z}) = 0$$

(3) $$\bar{V} + (-\bar{I}\bar{Z}) = 0$$

The interpretation placed upon the meaning of V and IZ leads to these three expressions, all giving the same final results. In the discussions in this chapter, the third method is used as the basis for solving the electrical circuits. The voltages, as measured by a voltmeter, are voltages across impedances or terminal voltages. The measurement of generated voltages cannot be made directly but must be computed from the meter readings and machine impedances. Figure 1–10 shows an electrical circuit having a source of electrical power and a sink (load). The terminal voltage V_{AC}, as indicated by a voltmeter, and the generated voltage E_{AC} are equal in magnitude but opposite in direction.

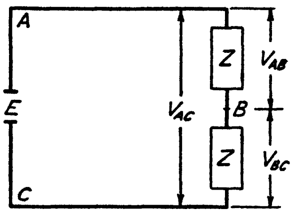


FIG. 1–10. Diagram showing the use of subscripts with voltage, current, and impedance: $\bar{V}_{AC} = \bar{V}_{AB} + \bar{V}_{BC}$.

2. Subscript Notation. If a certain current or voltage of a circuit is referred to, it is often very difficult to designate clearly which one is being considered. Circuit solutions become very involved because of this difficulty of identification. In the example (Fig. 1–10) the voltage V_{AB} is the voltage across the impedance between the points A and B on the diagram and the voltage V_{BC} is the voltage across the impedance between the points B and C. The voltage V_{AC} is the sum of V_{AB} and V_{BC} added vectorially.

$$\bar{V}_{AC} = \bar{V}_{AB} + \bar{V}_{BC}$$

and

$$\bar{V}_{CA} = \bar{V}_{CB} + \bar{V}_{BA}$$

and

$$\bar{V}_{CA} = -\bar{V}_{AC}$$

The voltage V_{AB} is a voltage drop caused by the current I_{AB} flowing through the impedance Z_{AB}, and the voltage V_{BC} is a voltage drop caused by the current I_{BC} flowing through the impedance Z_{BC}.

$$V_{AB} = I_{AB}Z_{AB} \quad \text{and} \quad \bar{V}_{AB} = \bar{I}_{AB}\bar{Z}_{AB}$$

$$V_{BC} = I_{BC}Z_{BC} \quad \text{and} \quad \bar{V}_{BC} = \bar{I}_{BC}\bar{Z}_{BC}$$

$$\bar{V}_{AC} = \bar{V}_{AB} + \bar{V}_{BC} = \bar{I}_{AB}\bar{Z}_{AB} + \bar{I}_{BC}\bar{Z}_{BC}$$

Then

$$\bar{V}_{AC} = \bar{I}_{AB}(\bar{Z}_{AB} + \bar{Z}_{BC}) = \bar{I}_{AC}\bar{Z}_{AC}$$

and

$$V_{AC} = I_{AC}Z_{AC}$$

If the generated voltage E_{AC} is used instead of the terminal voltage V_{AC}, the voltage equation becomes

$$-\bar{E}_{AC} = \bar{I}_{AC}\bar{Z}_{AC}$$

or

$$\bar{E}_{AC} = \bar{I}_{CA}\bar{Z}_{CA}$$

The equation stating Kirchhoff's Second Law for this simple circuit may be written:

For terminal voltage $\quad \bar{V}_{AC} + \bar{I}_{CA}\bar{Z}_{CA} = 0$

For generated voltage $\quad \bar{E}_{AC} - \bar{I}_{CA}\bar{Z}_{CA} = 0$

Since impedances are not vector quantities, the order of the subscripts used with them is not important. However, it is suggested that subscripts be used to designate definitely the impedance being considered. It is suggested that the same sequence of subscripts be used for impedance as for the current associated with the impedance in order to avoid confusion and possible error.

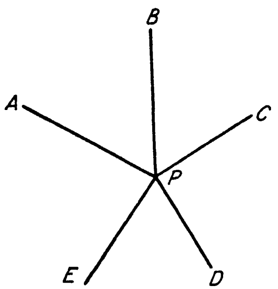


FIG. 2-10. Current relationships for any junction: $\bar{I}_{AP} + \bar{I}_{BP} + \bar{I}_{CP} + \bar{I}_{DP} + \bar{I}_{EP} = 0$.

In Fig. 2-10, the currents flowing in the leads terminate at the junction P. In general, these various currents establish current equilibrium at the point P. This means that the vector sum of the currents toward or away from the junction P is zero.

$$\bar{I}_{AP} + \bar{I}_{BP} + \bar{I}_{CP} + \bar{I}_{DP} + \bar{I}_{EP} = 0$$

or

$$\bar{I}_{PA} + \bar{I}_{PB} + \bar{I}_{PC} + \bar{I}_{PD} + \bar{I}_{PE} = 0$$

As previously stated, all the currents cannot actually flow toward the junction P, and the equation given above does not state that these currents are all flowing in the same direction. This relationship can be written

$$\bar{I}_{BP} + \bar{I}_{CP} + \bar{I}_{DP} + \bar{I}_{EP} = -\bar{I}_{AP} = \bar{I}_{PA}$$

and the equation has a different form, though its relationship is the same as that given in the previous equations. This junction equation can be written in many ways but the same relationship is expressed; namely, the vector sum of the currents toward or away from a junction equals zero.

3. Application of Kirchhoff's Laws: General. Sometimes it is difficult to decide whether to use Kirchhoff's Laws or Ohm's Law, especially if the circuit is comparatively simple. In the more complex circuits, Ohm's

Law is applicable to any part involving V, I, and Z of that part, whereas Kirchhoff's Laws are used for the individual loops or closed circuits. In solving a complicated electrical network involving several sources of power and many impedances in series-parallel combinations, the entire system should always be reduced to its simplest form before attempting to use Kirchhoff's Laws. Circuits including machines and associated equipment are not an exception to this rule, for any piece of electrical equipment operating at a definite set of conditions can be replaced in a circuit by equivalent values of R, $\mathcal{L}$, and C. For example, a d-c shunt motor can be replaced by the electrical circuit (Fig. 3–10a), and a transformer by the circuit (Fig. 3–10b). The substitution of equivalent circuits or impedances for machines simplifies the circuit solution, because the machine is then treated in the same way as any other impedance. It must be remembered that a complete solution can be obtained only when complete information is available for two of the three components (V, I, and Z) of the electrical circuit. This means, for currents or voltages, that the magnitude and direction must be available and for the impedance that the values of R, $\mathcal{L}$, and C, comprising the opposition, must be known.

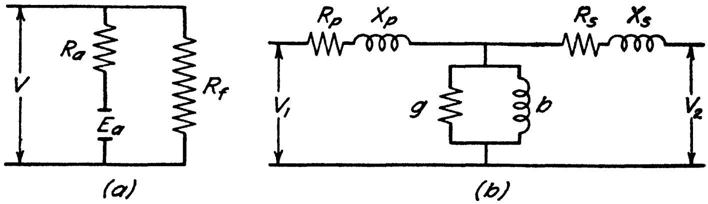


FIG. 3–10. (a) Equivalent circuit diagram of a shunt motor. (b) Equivalent circuit diagram of a transformer.

When the network has been simplified by combining the series impedances into a single impedance and combining the parallel impedances into a single impedance, and when information concerning the circuit parameters has been made available, Kirchhoff's Law equations can be written and a complete solution obtained.

4. Kirchhoff's Laws: Series Circuit. Figure 4–10a shows the circuit diagram of a circuit containing three different voltages connected in series and acting on the various impedances connected in combinations as given. By combining the impedances into a single equivalent impedance and by combining the voltages into a single voltage, the original circuit can be simplified into the circuit shown in Fig. 4–10b. The resultant voltage $\bar{V}_{MN}$ and the resultant impedance $\bar{Z}_0$ are the component parts used in the simple circuit. The total current in each part of the

circuit diagrams is the same and is found by using the diagram (Fig. 4–10*b*).

$$I = \frac{V_0}{Z_0}$$

If subscripts are used to identify definitely the component parts of this circuit (Fig. 4–10*b*), the expression for the voltage around the closed circuit becomes

$$\bar{V}_{MN} + \bar{I}_{RP}\bar{Z}_{RP} = 0 \quad \text{(Kirchhoff's Second Law)}$$

or

$$\bar{V}_{NM} + \bar{I}_{PR}\bar{Z}_{PR} = 0$$

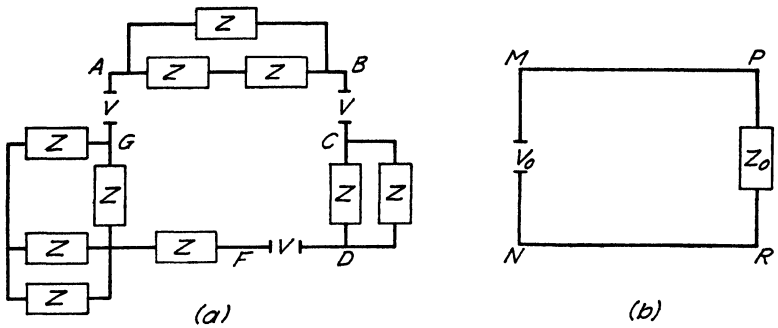


FIG. 4–10. (*a*) A circuit containing three voltages and ten impedances. (*b*) The same circuit reduced to one containing one voltage and one impedance.

One represents counterclockwise direction around the circuit, and the other represents clockwise direction. If the expression

$$\bar{V}_{NM} + \bar{I}_{PR}\bar{Z}_{PR} = 0$$

is solved,

$$\bar{I}_{PR} = \frac{-\bar{V}_{NM}}{\bar{Z}_{PR}} = \frac{\bar{V}_{MN}}{\bar{Z}_{PR}}$$

This current, expressed in complex form, depends upon the complex expressions for the source voltage and load impedance, and will never be more than 90° ahead or 90° behind the source voltage, because the maximum angular displacement which occurs for pure capacitive reactance and pure inductive reactance is 90°.

A more complex circuit having several sources of power is shown in Fig. 5–10*a* and the simplified circuit is shown in Fig. 5–10*b*. If the original circuit has several sources of power, a simple circuit, which contains only one source of power and one equivalent load, is difficult to obtain, because the series-parallel grouping of the impedances is gen-

erally interdependent and influenced by the various power sources. The current flowing in one impedance $\bar{Z}$ can be calculated from the voltage equation for the closed loop containing that impedance.

This equation can be written

$$\Sigma\bar{V} + \bar{I}\bar{Z} = 0$$

$$\bar{I} = -\frac{\Sigma\bar{V}}{\bar{Z}}$$

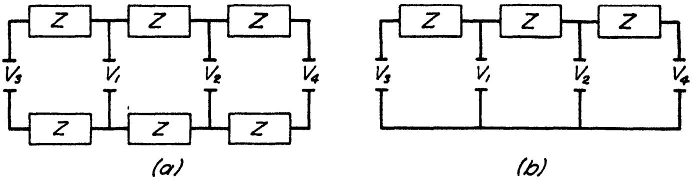


FIG. 5–10. (a) A section of a network containing four sources of power. (b) Section reduced to a simplified circuit.

Example a. A load impedance $(2.8 + j3.8)$ is connected to a source of power $(80 + j60)$ by leads, each having an impedance of $(0.1 + j0.1)$. Determine the value of the current in numerical and complex form.

Figure 6–10 shows the circuit diagram. The voltage equation is

$$\bar{V}_{DA} + \bar{I}_{AB}(0.1 + j0.1) + \bar{I}_{BC}(2.8 + j3.8) + \bar{I}_{CD}(0.1 + j0.1) = 0$$

$(\bar{I}_{AB} = \bar{I}_{BC} = \bar{I}_{CD})$

$$\bar{V}_{DA} + \bar{I}_{AB}(3 + j4) = 0$$

$$\bar{I}_{AB} = \frac{80 + j60}{3 + j4} = 19.2 - j5.6$$

$$I_{AB} = 20 \text{ amp}$$

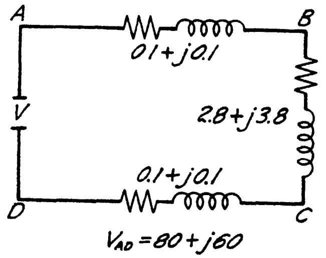


FIG. 6–10. Circuit diagram for Example *a*.

Example b. Given a closed loop with four terminal voltages and four impedances. Write the voltage equation for the loop, solve for the current, and draw the vector diagram.

$\bar{V}_{AB} = 100 + j0$ $\quad\bar{V}_{CD} = -50 + j50$ $\quad\bar{V}_{EF} = 75 - j10$ $\quad\bar{V}_{GH} = 75 + j60$

$\bar{Z}_{BC} = 3 - j4$ $\quad\bar{Z}_{DE} = 8 + j12$ $\quad\bar{Z}_{FG} = 2 - j12$ $\quad\bar{Z}_{HA} = 3 - j8$

Figure 7–10 shows the circuit diagram and the vector diagram. Voltage equation in order of subscript letters:

$$\bar{V}_{AB} + \bar{I}_{BC}\bar{Z}_{BC} + \bar{V}_{CD} + \bar{I}_{DE}\bar{Z}_{DE} + \bar{V}_{EF} + \bar{I}_{FG}\bar{Z}_{FG} + \bar{V}_{GH} + \bar{I}_{HA}\bar{Z}_{HA} = 0$$

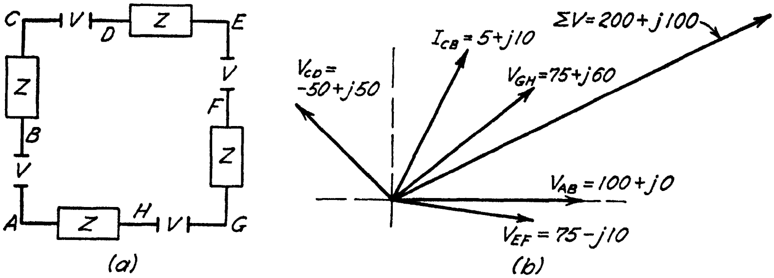


FIG. 7–10. Circuit diagram and vector diagram for Example b.

but

$$\bar{I}_{BC} = \bar{I}_{DE} = \bar{I}_{FG} = \bar{I}_{HA}$$

$$\bar{I}_{CB} = \frac{\bar{V}_{AB} + \bar{V}_{CD} + \bar{V}_{EF} + \bar{V}_{GH}}{\bar{Z}_{BC} + \bar{Z}_{DE} + \bar{Z}_{FG} + \bar{Z}_{GA}}$$

$$\bar{I}_{CB} = \frac{(100 + j0) + (-50 + j50) + (75 - j10) + (75 + j60)}{(3 - j4) + (8 + j12) + (2 - j12) + (3 - j8)}$$

$$\bar{I}_{CB} = \frac{200 + j100}{16 - j12} = 5 + j10$$

$$\bar{I}_{BC} = -5 - j10$$

$$\bar{I}_{CB} = \sqrt{125} = 11.1 \text{ amp}$$

5. Kirchhoff's Laws: Parallel Circuit. Parallel circuits consist of several impedances interconnected to the same sources. When the circuit network contains several sources of power, both Kirchhoff's Laws must be used because there are more unknown currents than closed loops, and the necessary additional equations must be current equations. It is always advisable to write as many voltage equations as possible and complete the required number of equations from the current relationships. Before attempting to solve a network circuit, the number of unknown currents must be determined. Figure 8–10 shows a

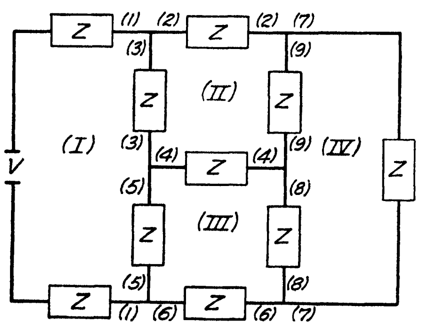


FIG. 8–10. Circuit diagram; nine unknown currents.

circuit diagram of impedances in parallel and in series. The unknown currents have been numbered. To solve this circuit, nine independent (prime) equations for voltage or current relationships are required. The prime equations of the electrical circuit are voltage equations for independent loops and current equations for independent junctions. In solving this example (Fig. 8–10), nine equations are necessary: four voltage equations for the loops I, II, III, and IV and five current equations at any five of the six current junctions. It is only possible to write four independent voltage equations (one for each bounded area), and the remaining equations must be current equations.

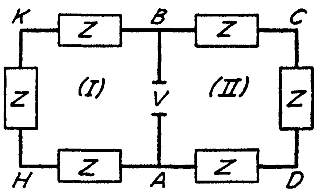


FIG. 9–10. One power source supplying two separate impedance branches.

When a single power source supplies two independent loads, the two currents flowing to the two loads must come from the same source; and the total source current will be the vector sum of the two individual load currents. Consider the circuit shown in Fig. 9–10. Loops I and II are independent of each other, and the current of each loop depends upon its own impedance and the source voltage.*

The voltage equations for each loop, going clockwise around the loop, are:

Loop I $\quad \bar{V}_{BA} + \bar{I}_{AH}\bar{Z}_{AH} + \bar{I}_{HK}\bar{Z}_{HK} + \bar{I}_{KB}\bar{Z}_{KB} = 0$

$$\bar{I}_{AH} = \bar{I}_{HK} = \bar{I}_{KB}$$

$$\bar{V}_{BA} + \bar{I}_{AH}(\bar{Z}_{AH} + \bar{Z}_{HK} + \bar{Z}_{KB}) = 0$$

$$\bar{I}_{AH} = \frac{-\bar{V}_{BA}}{\bar{Z}_{AH} + \bar{Z}_{HK} + \bar{Z}_{KB}} = \frac{\bar{V}_{AB}}{\bar{Z}_{AH} + \bar{Z}_{HK} + \bar{Z}_{KB}}$$

Loop II $\quad \bar{V}_{AB} + \bar{I}_{BC}\bar{Z}_{BC} + \bar{I}_{CD}\bar{Z}_{CD} + \bar{I}_{DA}\bar{Z}_{DA} = 0$

$$\bar{I}_{BC} = \bar{I}_{CD} = \bar{I}_{DA}$$

$$\bar{V}_{AB} + \bar{I}_{BC}(\bar{Z}_{BC} + \bar{Z}_{CD} + \bar{Z}_{DA}) = 0$$

$$\bar{I}_{BC} = \frac{-\bar{V}_{AB}}{\bar{Z}_{BC} + \bar{Z}_{CD} + \bar{Z}_{DA}} = \frac{\bar{V}_{BA}}{\bar{Z}_{BC} + \bar{Z}_{CD} + \bar{Z}_{DA}}$$

The current supplied by the source V_{AB} is the vector sum of the two currents furnished to loops *AHKB* and *BCDA*. In this network, there

* Figure 9–10 shows the simplest parallel circuit. Two independent simple circuits are connected to the same source voltage and have currents in each which are easily determined.

are three different currents, one for loop $AHKB$, one for loop $BCDA$, and the total current. The third equation, therefore, is a current equation for a junction point. Kirchhoff's Second Law for junction "B" is

$$\bar{I}_{AB} + \bar{I}_{CB} + \bar{I}_{KB} = 0$$

The three equations necessary to give a complete solution are

(1) Voltage equation loop $AHKB$

(2) Voltage equation loop $BCDA$

(3) Current equation point B or point A

Example c. Write the equations required to solve the circuit shown in Fig. 10–10.

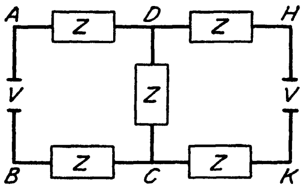


FIG. 10–10. Circuit diagram for Example c.

Since there are three unknown currents, three prime equations are necessary; that is, two voltage equations and one current equation.

Clockwise direction,
around loop $BADCB$: $\bar{V}_{BA} + \bar{I}_{AD}\bar{Z}_{AD} + \bar{I}_{DC}\bar{Z}_{DC} + \bar{I}_{CB}\bar{Z}_{CB} = 0$

(1) $\bar{I}_{AD} = \bar{I}_{CB}$: $\bar{V}_{BA} + \bar{I}_{AD}(\bar{Z}_{AD} + \bar{Z}_{CB}) + \bar{I}_{DC}\bar{Z}_{DC} = 0$

around loop $HKCDH$:

$\bar{V}_{HK} + \bar{I}_{KC}\bar{Z}_{KC} + \bar{I}_{CD}\bar{Z}_{CD} + \bar{I}_{DH}\bar{Z}_{DH} = 0$

(2) $\bar{I}_{DH} = \bar{I}_{KC}$: $\bar{V}_{HK} + \bar{I}_{DH}(\bar{Z}_{KC} + \bar{Z}_{DH}) + \bar{I}_{CD}\bar{Z}_{CD} = 0$

(3) Junction D: $\bar{I}_{AD} + \bar{I}_{HD} + \bar{I}_{CD} = 0$

If complex values are assigned to the various voltages and impedances in the example just given, the three equations may be solved and the three currents determined in both complex and numerical form.

6. Series-Parallel Circuits. If the circuit of Fig. 9–10 is changed to contain an impedance in the source lead (Fig. 11–10a), the solution, though more complicated, is obtained by the same method of attack.

Figure 11–10b is an equivalent diagram for the circuit shown in Fig. 11–10a. Although this circuit may be reduced to a single impedance across the source voltage, the circuit will be discussed as shown in Fig. 11–10a. The advantage of discussing this type of circuit is that it

may be solved by another method and the two results compared. In the circuit as given, there are three unknown currents and three independent equations are necessary. The equations using clockwise direction around the voltage loops are

(1) Loop $XBAHK$: $\quad \bar{V}_{BA} + \bar{I}_{KX}(\bar{Z}_{AH} + \bar{Z}_{HK} + \bar{Z}_{KX}) + \bar{I}_{XB}\bar{Z}_{XB} = 0$

$$\bar{I}_{AH} = \bar{I}_{HK} = \bar{I}_{KX}$$

(2) Loop $XCDAB$: $\quad \bar{V}_{AB} + \bar{I}_{BX}\bar{Z}_{BX} + \bar{I}_{XC}(\bar{Z}_{XC} + \bar{Z}_{CD} + \bar{Z}_{DA}) = 0$

$$\bar{I}_{XC} = \bar{I}_{CD} = \bar{I}_{DA}$$

(3) Junction X: $\quad \bar{I}_{BX} + \bar{I}_{KX} + \bar{I}_{CX} = 0$

These three equations contain the three unknown currents and the value (vector and absolute) of each can be found.

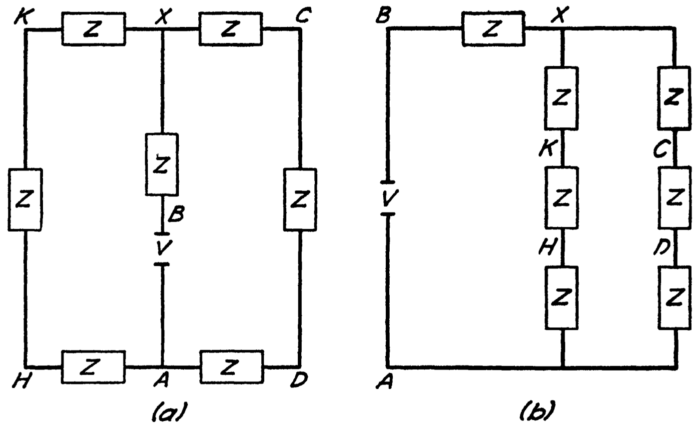


FIG. 11–10. A network with one source and two parallel branches of impedance.

Example d. Assume that Fig. 11–10 represents a generator supplying power to two loads, $\bar{Z}_{KH}$ and $\bar{Z}_{CD}$, over wires having the impedances indicated. Determine the current in each branch and the total current if the following complex expressions are assigned to the voltage and impedances. Draw a vector diagram showing the voltage V_{AB} and the three currents.

$$\bar{V}_{AB} = 100 + j0$$
$$\bar{Z}_{BX} = \bar{Z}_{KX} = \bar{Z}_{CX} = 1 + j1$$
$$\bar{Z}_{HA} = \bar{Z}_{DA} = 1 + j1$$
$$\bar{Z}_{KH} = 3 + j4$$
$$\bar{Z}_{CD} = 8 - j6$$

Three equations are necessary.

Loop $XBAHKX$: $\quad \bar{V}_{BA} + \bar{I}_{KX}(\bar{Z}_{AH} + \bar{Z}_{HK} + \bar{Z}_{KX}) + \bar{I}_{XB}\bar{Z}_{XB} = 0$

Substituting the values,

$$(-100 + j0) + \bar{I}_{KX}[(1 + j1) + (3 + j4) + (1 + j1)] + \bar{I}_{XB}(1 + j1) = 0$$

(1) $$(-100 + j0) + \bar{I}_{KX}(5 + j6) + \bar{I}_{XB}(1 + j1) = 0$$

Loop $XCDABX$: $\bar{V}_{AB} + \bar{I}_{BX}\bar{Z}_{BX} + \bar{I}_{XC}(\bar{Z}_{XC} + \bar{Z}_{CD} + \bar{Z}_{DA}) = 0$

Substituting the values,

$$(100 + j0) + \bar{I}_{BX}(1 + j1) + \bar{I}_{XC}[(1 + j1) + (8 - j6) + (1 + j1)] = 0$$

(2) $$(100 + j0) + \bar{I}_{BX}(1 + j1) + \bar{I}_{XC}(10 - j4) = 0$$

(3) Junction X: $\bar{I}_{BX} + \bar{I}_{KX} + \bar{I}_{CX} = 0$

Since $\bar{I}_{BX}$ appears in both voltage equations, substitute for $\bar{I}_{BX}$ and the two voltage equations will contain only two unknown currents, $\bar{I}_{KX}$ and $\bar{I}_{CX}$.

Substituting in (1) for $\bar{I}_{XB} = \bar{I}_{KX} + \bar{I}_{CX}$,

$$(-100 + j0) + \bar{I}_{KX}(5 + j6) + \bar{I}_{KX}(1 + j1) + \bar{I}_{CX}(1 + j1) = 0$$

(1*a*) $$(-100 + j0) + \bar{I}_{KX}(6 + j7) + \bar{I}_{CX}(1 + j1) = 0$$

Substituting in (2) for $\bar{I}_{BX} = -\bar{I}_{KX} - \bar{I}_{CX}$,

$$(100 + j0) + (-\bar{I}_{KX})(1 + j1) + (-\bar{I}_{CX})(1 + j1) + \bar{I}_{XC}(10 - j4) = 0$$

Changing $+\bar{I}_{XC}$ to $-\bar{I}_{CX}$,

$$(100 + j0) - \bar{I}_{KX}(1 + j1) - \bar{I}_{CX}(1 + j1) - \bar{I}_{CX}(10 - j4) = 0$$

(2*a*) $$(100 + j0) - \bar{I}_{KX}(1 + j1) - \bar{I}_{CX}(11 - j3) = 0$$

The two equations containing the two unknowns $\bar{I}_{KX}$ and $\bar{I}_{CX}$ are

(1*b*) $$(-100 + j0) + \bar{I}_{KX}(6 + j7) + \bar{I}_{CX}(1 + j1) = 0$$

and

(2*b*) $$(100 + j0) - \bar{I}_{KX}(1 + j1) - \bar{I}_{CX}(11 - j3) = 0$$

Eliminating $\bar{I}_{KX}$ from both equations by cross-multiplying the coefficients of $\bar{I}_{KX}$,

(1*c*) Multiply (1*b*) by $(1 + j1)$:

$$(-100 - j100) + \bar{I}_{KX}(6 + j7)(1 + j1) + \bar{I}_{CX}(0 + j2) = 0$$

(2*c*) Multiply (2*b*) by $(6 + j7)$:

$$(600 + j700) - \bar{I}_{KX}(6 + j7)(1 + j1) - \bar{I}_{CX}(87 + j59) = 0$$

(1*c* + 2*c*) $$(500 + j600) - \bar{I}_{CX}(87 + j57) = 0$$

Solving,

$$\bar{I}_{CX} = \frac{500 + j600}{87 + j57} = +7.182 + j2.190$$

Solving for $\bar{I}_{KX}$ in equation (2b),

$$(100 + j0) - \bar{I}_{KX}(1 + j1) - (+7.182 + j2.190)(11 - j3) = 0$$

$$(100 + j0) - \bar{I}_{KX}(1 + j1) - (+85.572 + j2.544) = 0$$

$$(14.428 - j2.544) - \bar{I}_{KX}(1 + j1) = 0$$

$$\bar{I}_{KX} = \frac{14.428 - j2.544}{1 + j1} = +5.942 - j8.486$$

$$\bar{I}_{CX} + \bar{I}_{BX} + \bar{I}_{KX} = 0$$

$$\bar{I}_{BX} = -\bar{I}_{CX} - \bar{I}_{KX} = -(7.182 + j2.190) - (+5.942 - j8.486)$$

$$\bar{I}_{BX} = -13.124 + j6.296$$

Figure 12–10 shows the vector diagram.

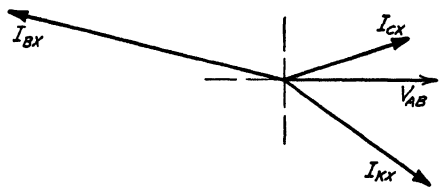


FIG. 12–10. Vector diagram for the network in Fig. 11–10.

7. General Circuit Solution. To aid in obtaining a solution of a complicated network problem, the following rules are offered.

(*a*) The circuit should be reduced to its simplest form. Impedances in parallel should be reduced to one equivalent impedance, and impedances in series to an equivalent impedance.

(*b*) The circuit components (voltages, currents, and impedances) should be designated by subscript numbers or letters.

(*c*) The number of unknowns should be determined. This determines the number of prime equations necessary to obtain a complete solution.

(*d*) Write as many independent voltage (emf) equations as possible. All independent loops or bounded areas are considered the basis for these equations. (Independent loops or areas are those which cannot be obtained by adding or subtracting other loops or areas.)

The additional equations needed can be obtained by writing independent current equations, junction points which are common to the various loops or areas being used.

(*e*) Solve the equations as simultaneous equations. All complex expressions obey the simple laws of algebra and can be solved by algebraic methods.

(*f*) The solution in subscripts and complex form gives the magnitude and direction of the unknown quantities.

8. Tabular Form for Solving Equations. The following method uses detached coefficients, and these are tabulated in a definite sequence. The number of unknown currents is determined and each placed at the head of a column. The coefficients are listed under the proper heading as the prime equations are written. The column marked "check" is a column listing the operations on the coefficients. After the prime equations are listed, the check column is used to determine whether or not a mistake has been made in the mathematical manipulations. The "check" column value is always the sum of the coefficients and constant for that equation.

Example e (Fig. 13–10). Determine the current flowing in each part of the circuit for the values of voltages given.

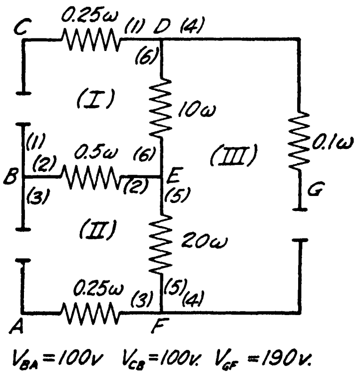


FIG. 13–10. Example of a d-c network.

By the method of numbering the currents at the junctions, six unknown currents exist and six equations are needed. Since this is a d-c circuit, algebraic addition and symbolic addition give the same results. The operations or procedure in reducing the six equations is indicated under the column marked "operation."

(1) Loop *ABEFA*: $-100 + 0.5I_{BE} + 20I_{EF} + 0.25I_{FA} = 0$

(2) Loop *BCDEB*: $-100 + 0.25I_{CD} + 10I_{DE} + 0.5I_{EB} = 0$

(3) Loop *DEFGD*: $+10I_{DE} + 20I_{EF} - 190 + 0.1I_{GD} = 0$

(4) Junction *D*: $I_{CD} + I_{ED} + I_{GD} = 0$

(5) Junction *E*: $I_{DE} + I_{BE} + I_{FE} = 0$

(6) Junction *F*: $I_{EF} + I_{AF} + I_{DG} = 0$

In tabular form the equations appear as

No.	Operation	Six Unknown Currents						K	Check
		I_{CD}	I_{ED}	I_{GD}	I_{BE}	I_{FE}	I_{AF}		
1		0	0	0	0.5	−20	−0.25	−100	−119.75
2		0.25	−10	0	−0.5	0	0	−100	−110.25
3		0	−10	0.1	0	−20	0	−190	−219.90
4		1	1	1	0	0	0	0	3
5		0	−1	0	1	1	0	0	1
6		0	0	−1	0	−1	1	0	−1
7	4 × No. 2	1	−40	0	−2	0	0	−400	−441
8	No. 4 − No. 7	0	41	1	2	0	0	400	444
9	4 × No. 1	0	0	0	2	−80	−1	−400	−479
10	No. 6 + No. 9	0	0	−1	2	−81	0	−400	−480
11	10 × No. 3	0	−100	1	0	−200	0	−1900	−2199
12	No. 8 + No. 10	0	41	0	4	−81	0	0	−36
13	No. 10 + No. 11	0	−100	0	2	−281	0	−2300	−2679
14	No. 12 ÷ 4	0	10.25	0	1	−20.25	0	0	−9
15	No. 13 ÷ 2	0	−50	0	1	−140.5	0	−1150	−1339.5
16	No. 14 − No. 5	0	11.25	0	0	−21.25	0	0	−10
17	No. 15 − No. 5	0	−49	0	0	−141.5	0	−1150	−1340.5

From equation 16,

$$11.25I_{ED} = 21.25I_{FE}$$

and

$$I_{ED} = \frac{21.25}{11.25} I_{FE}$$

Substituting in 17,

$$-49I_{ED} - 141.5I_{FE} - 1150 = 0$$

$$-49\left(\frac{21.25}{11.25}\right) I_{FE} - 141.5I_{FE} - 1150 = 0$$

$$I_{FE} = -4.913363 \text{ amp}$$

and

$$I_{ED} = -9.280796 \text{ amp}$$

Substituting in 15,

$$-50I_{ED} + I_{BE} - 140.5I_{FE} - 1150 = 0$$

$$I_{BE} = 50(-9.280796) + 140.5(-4.913363) + 1150$$

$$I_{BE} = -4.3673 \text{ amp}$$

Substituting in 11,

$$-100I_{ED} + I_{GD} - 200I_{FE} - 1900 = 0$$

$$I_{GD} = 1900 + 100(-9.280796) + 200(-4.913363)$$

$$I_{GD} = -10.7522 \text{ amp}$$

Substituting in 9,

$$2I_{BE} - 80I_{FE} - I_{AF} - 400 = 0$$

$$I_{AF} = -400 + 2(-4.3673) - 80(-4.913363)$$

$$I_{AF} = -15.6656 \text{ amp}$$

Substituting in 7,

$$I_{CD} - 40I_{ED} - 2I_{BE} - 400 = 0$$

$$I_{CD} = 400 + 40(-9.280796) + 2(-4.3673)$$

$$I_{CD} = 20.0336 \text{ amp}$$

$I_{CD} = 20.0336$ amp	$I_{DE} = 9.2808$ amp
$I_{BE} = -4.3673$ amp	$I_{EF} = 4.9134$ amp
$I_{AF} = -15.6656$ amp	$I_{DG} = 10.7522$ amp

Example f. For Fig. 14–10, determine the currents in each part of the circuit in complex and numerical form. Draw the vector diagram.

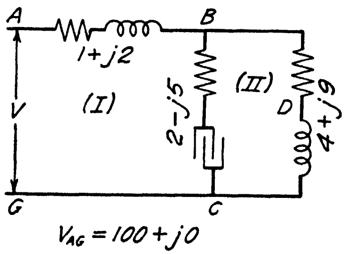


FIG. 14–10. Circuit diagram for Example *f*.

This circuit can be reduced to a single impedance across the source voltage and the total current determined. After the total current has been determined, the current in the two parallel branches can be computed. Instead of using this simple method of attack, the problem will be solved by the use of Kirchhoff's Laws, and the solution will illustrate the use of symbolic coefficients.

In this example, there are three unknown currents and the three prime equations are

Clockwise around loop *GABCG*:

(1) $$\bar{V}_{GA} + \bar{I}_{AB}(1 + j2) + \bar{I}_{BC}(2 - j5) = 0$$

$$-(100 + j0) + \bar{I}_{AB}(1 + j2) + \bar{I}_{BC}(2 - j5) = 0$$

Counterclockwise around loop *BCDB*:

(2) $$+\bar{I}_{BC}(2 - j5) + \bar{I}_{CDB}(4 + j9) = 0$$

Junction *B*:

(3) $$\bar{I}_{AB} + \bar{I}_{CB} + \bar{I}_{CDB} = 0$$

Assume that the currents to be determined are $\bar{I}_{AB}$, $\bar{I}_{BC}$, and $\bar{I}_{BDC}$.

In tabular form, the equations appear as

No.	Operation	Three Unknown Currents			K'	Check
		$\bar{I}_{AB}$	$\bar{I}_{BC}$	$\bar{I}_{BDC}$		
1		$(1 + j2)$	$(2 - j5)$	0	$-(100 + j0)$	$-(97 + j3)$
2		0	$(2 - j5)$	$-(4 + j9)$	0	$-(2 + j14)$
3		1	-1	-1	0	-1[1]
4	$(1 + j2) \times$ No. 3	$(1 + j2)$	$-(1 + j2)$	$-(1 + j2)$	0	$-1 - j2$
5	No. 4 − No. 1	0	$-(3 - j3)$	$(-1 - j2)$	$100 + j0$	$96 + j1$
6	$(3 - j3) \times$ No. 2	0	$-(9 + j21)$	$-(39 + j15)$	0	$-(48 + j36)$
7	$(2 - j5) \times$ No. 5	0	$(9 + j21)$	$(-12 + j1)$	$200 - j500$	$197 - j478$
8	No. 6 + No. 7	0	0	$-(51 + j14)$	$200 - j500$	$149 - j514$

[1] The coefficients of the unknown currents may be written either positive or negative. In this solution, both have been used

From equation 8,

$$\bar{I}_{BDC} = \frac{200 - j500}{51 + j14} = 1.15 - j10.11$$

$$I_{BDC} = 10.15 \text{ amp}$$

Substituting in 2,

$$(2 - j5)\bar{I}_{BC} - (4 + j9)\bar{I}_{BDC} = 0$$

$$\bar{I}_{BC} = \frac{(4 + j9)(1.15 - j10.11)}{2 - j5} = 11.79 + j14.42$$

$$I_{BC} = 18.61 \text{ amp}$$

Substituting in 3,

$$\bar{I}_{AB} = \bar{I}_{BC} + \bar{I}_{BDC}$$

$$\bar{I}_{AB} = (11.79 + j14.42) + (1.15 - j10.11)$$

$$\bar{I}_{AB} = 12.94 + j4.31$$

$$I_{AB} = 13.6 \text{ amp}$$

$$\bar{I}_{AB} = 12.94 + j4.31 \qquad I_{AB} = 13.6 \text{ amp}$$

$$\bar{I}_{BC} = 11.79 + j14.42 \qquad I_{BC} = 18.61 \text{ amp}$$

$$\bar{I}_{BDC} = 1.15 - j10.11 \qquad I_{BDC} = 10.15 \text{ amp}$$

$$\bar{V}_{AB} = \bar{I}_{AB}\bar{Z}_{AB} = (12.94 + j4.31)(1 + j2) = 4.32 + j30.19$$

$$V_{AB} = 30.4 \text{ volts}$$

$$\bar{V}_{BC} = \bar{I}_{BC}\bar{Z}_{BC} = (11.79 + j14.42)(2 - j5) = 95.68 - j30.11$$

$$V_{BC} = 100.1 \text{ volts}$$

$$\bar{V}_{BDC} = \bar{I}_{BDC}\bar{Z}_{BDC} = (1.15 - j10.11)(4 + j9) = 95.59 - j30.09$$

$$V_{BDC} = 100 \text{ volts}$$

The vector diagram is shown in Fig. 15–10. It will be noted that the voltages V_{BC} and V_{BDC} are greater than the impressed voltage V_{AG}. However, the vector summation of the voltages around any closed loop total zero.

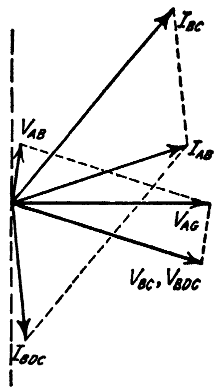


FIG. 15–10. Vector diagram for Fig. 14–10.

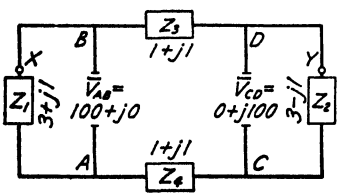
PROBLEMS

1–10. Five conductors are joined at a junction N. $\bar{I}_{AN} = 10 + j10$; $\bar{I}_{NB} = 8 - j6$; $\bar{I}_{NC} = 12 + j16$; $\bar{I}_{DN} = 5 + j5$. Determine $\bar{I}_{EN}$ and the magnitude of the current in each conductor.

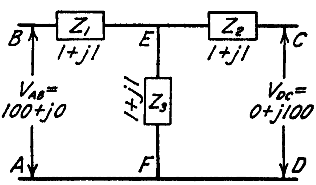
2–10. Assume all impedances for the circuit diagram Fig. 4–10*a* (p. 180) to be $2 + j2$. $\bar{V}_{AG} = 100 + j0$; $\bar{V}_{FD} = 0 - j100$; $\bar{V}_{BC} = 0 - j100$. Determine $\bar{I}_{DC}$.

3–10. A power source consists of two sources, $\bar{V}_{AB} = 100 + j0$ and $\bar{V}_{CB} = 0 + j100$, connected in series. Three impedances, $\bar{Z}_1 = 5 + j0$, $\bar{Z}_2 = 3 + j4$, $\bar{Z}_3 = 0 + j2$, are connected in series across the source $\bar{V}_{AC}$. Determine the current and voltage across each impedance in complex form and numerically.

4–10. In the circuit in the figure determine the currents in the sources $\bar{V}_{AB}$ and $\bar{V}_{CD}$.



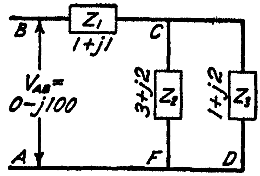
PROB. 4–10.



PROB. 5–10.

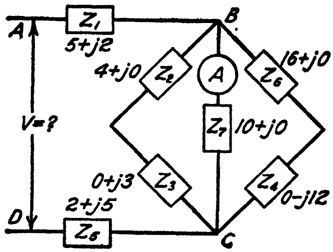
5–10. Determine the currents (complex form and magnitude) in each impedance for the circuit shown in the figure. Draw a complete vector diagram for the circuit.

6–10. Write the equations necessary to solve the circuit of the figure. Simplify the circuit and solve for all currents and voltages. Draw a complete vector diagram. How much current will flow in each opposition if the same value of d-c voltage is used at the supply?



PROB. 6–10.

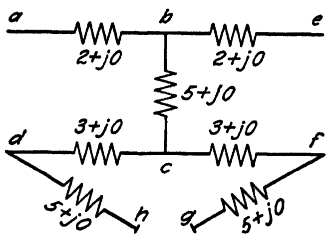
7–10. The ammeter (A) of the figure reads 10 amp when connected to an a-c voltage V_{AD}. What is the magnitude of this voltage?



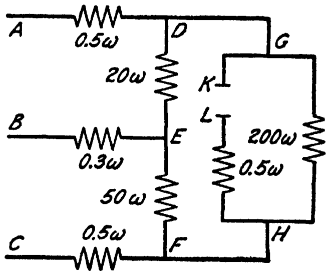
PROB. 7–10.

8–10. A load is supplied power from a 250-volt a-c source over a two-wire circuit. The impedance of each conductor is $0.1 + j0.5$. The load is inductive and requires 7500 watts. The line current is 50 amp. Determine the voltage across the load.

9–10. Solve for all currents in the circuit shown in the figure. $\bar{V}_{da} = 100 + j0$, $\bar{V}_{fe} = 100 + j0$, $\bar{V}_{gh} = 100 + j0$.



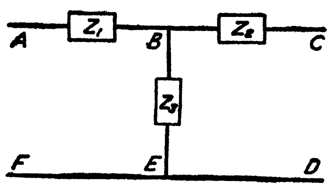
PROB. 9–10.



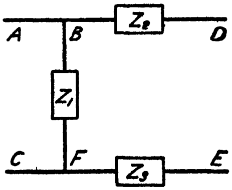
PROB. 10–10.

10–10. The figure shows the circuit diagram of a d-c distribution system. Solve for all currents. $\bar{V}_{CB} = 100$ volts, $V_{BA} = 100$ volts, $V_{LK} = 150$ volts.

11–10. For the circuit shown in the figure, $\bar{Z}_1 = 5 + j0$, $\bar{Z}_2 = 0 - j5$, $\bar{Z}_3 = 1 + j3$, $\bar{V}_{AF} = 50 + j0$, $\bar{V}_{CD} = 40 + j20$, and $\bar{I}_{BE} = 10 + j0$. Find $\bar{I}_{AB}$ and $\bar{I}_{CB}$.



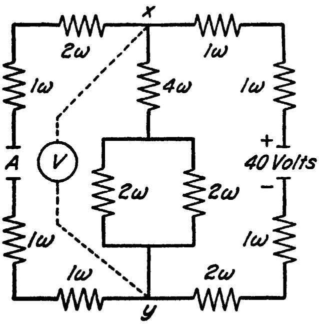
PROB. 11–10.



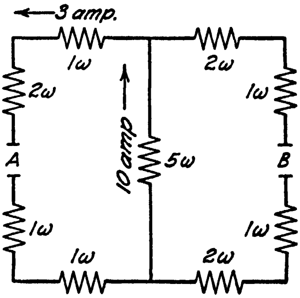
Prob. 12–10.

12–10. Determine the complex expression for each current in the circuit shown in the figure. $\bar{V}_{AC} = 80 + j0$, $\bar{V}_{DE} = 80 + j60$, $\bar{Z}_1 = 8 + j0$, $\bar{Z}_2 = 0 + j9$, and $\bar{Z}_3 = 0 + j3$.

13–10. In the figure the voltmeter (V) reads 25 volts with (X) the positive terminal. Indicate the polarity and value of source A and the value and direction of currents through each source.



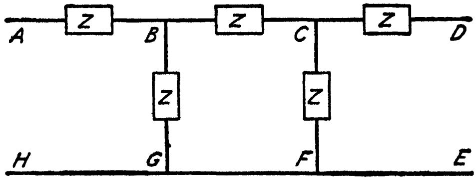
Prob. 13–10.



Prob. 14–10.

14–10. For the d-c system indicated in the figure the known currents and their directions are shown. Determine the value of voltage and polarity for the sources A and B.

15–10. The figure shows the circuit diagram for a system containing impedances and two voltages. The known values are $\bar{Z}_{AB} = 2 + j1$, $\bar{Z}_{BC} = 1 - j3$, $\bar{Z}_{CD} = 3 + j1$, $\bar{Z}_{BG} = 8 + j6$, $\bar{Z}_{CF} = 4 + j0$, $\bar{V}_{AH} = 120 + j10$, $\bar{I}_{AB} = 10 + j0$. Find $\bar{V}_{DE}$.



Prob. 15–10.

CHAPTER 11

POWER AND ENERGY: SINGLE-PHASE ALTERNATING- AND DIRECT-CURRENT POWER

The value of electrical generation lies in the ease of conversion of power for doing work, and it is because of this fact that rapid strides have been made in the economical production of electrical energy. The principal advantage of electricity for industrial purposes is the ease with which it may be transported. Compactness and safety of power equipment and unusual cleanliness are other factors influential in favor of the general adoption of electricity. Civilization has advanced with the availability of abundant and cheap power; therefore, it is natural that electricity, being both abundant and cheap, has in a relatively short time forged to the front in the industrial world.

The distribution of electrical power has been a factor in the development of fields other than electrical engineering, for electrical energy has become a commodity of basic consideration in the economic structure of civilization; also, the interest of legislative, legal, and commercial groups has been involved, for the existence of these groups has come to be dependent upon the generation and transmission of invisible forces. Therefore, the study of electrical power is not complete without an appreciation of the secondary problems involved.

1. Power and Energy. Power and energy differ only in the time element involved. If power is expended over a period of time, the total energy will be the product of the average power and the time, the units used depending upon the quantity measured. When power and energy are considered, there is a marked similarity between the various branches of engineering. In comparing the work involved in hydraulics, mechanics, and electrical engineering, it is found that the energy consumed in any operation depends upon the magnitude of the quantities under consideration and the opposition overcome. Expressions for energy are

$$\text{Mechanical energy} = kFs$$

$$\text{Hydraulic energy} = kQh$$

$$\text{Electrical energy} = kQV$$

where k is the proportionality factor (reduces to unity when the definitions are properly chosen), F is force, s is distance, Q is quantity of water or electricity, h is head or pressure, and V is potential difference or pressure.

Energy may be either useful (convertible) or of such a nature that it is inaccessible. In all conversions, some of the energy passes into an inaccessible form. Energy, therefore, is of two levels: that which lies above and that which lies below the useful datum. The energy below this datum cannot be converted and is continually increasing. In other words, since energy is never converted at 100 per cent efficiency, the datum is being raised and all energy is being leveled to the region of non-convertibility. It is only the convertible energy which is of interest to the engineer.

Power is the rate of doing work or the rate at which energy is transformed. To change the energy expressions, kFs, kQh, and kQV to power expressions, each is divided by time. These formulas then take the form:

$$\text{Mechanical} \quad P = k\frac{Fs}{t} = kFv$$

$$\text{Hydraulic} \quad P = k\frac{Qh}{t} = kqh$$

$$\text{Electrical} \quad P = k\frac{QV}{t} = kIV$$

where P is the power, v is the velocity, q is the quantity of water per second, and I is the current or coulombs per second. In each instance the power is the product of the rate of flow and the potential difference involved.

Power ratings are more commonly used, because interest lies in the rate at which energy is consumed and, when it is desired to determine the energy, it is necessary only to multiply the power by the time. All installations are rated on a power basis, but the consumer is charged for service on the basis of energy delivered.

2. Electrical Power. Power is a *scalar quantity* and is never considered in a *vector sense*. Special consideration must be given the power expression in dealing with alternating voltages and currents. Though there are some special expressions for power in a-c systems, *the power is always added algebraically*, and the resistance or conductance determines the power consumption in either the a-c or the d-c system. Power considerations always act as a final check on either a-c or d-c computations, because the *summation of the individual amounts of power dissipated at the*

sinks must equal the power of the sources. Either the sinks or the sources may be considered positive or negative, but it is necessary to have them relatively different in sign so that the algebraic summation is zero.

3. Instantaneous and Average Power. Instantaneous power is expressed by

$$p = vi$$

where p is the instantaneous power, v is the voltage, and i is the current at the same instant. This form, previously given in equation (a–4) Chapter 4, is true for alternating, direct, or periodic current. In either a-c or d-c circuits the average power is more important.

When

$$v = V_m \sin (\omega t \pm \theta)$$

$$i = I_m \sin \omega t$$

$$\begin{aligned}
p &= V_m I_m \sin \omega t \sin (\omega t \pm \theta) \\
&= V_m I_m \sin \omega t (\sin \omega t \cos \theta \pm \cos \omega t \sin \theta) \\
&= V_m I_m (\sin^2 \omega t \cos \theta \pm \sin \omega t \cos \omega t \sin \theta) \\
&= V_m I_m \left[\left(\frac{1 - \cos 2\omega t}{2} \cos \theta \right) \pm \left(\frac{\sin 2\omega t}{2} \sin \theta \right) \right] \\
&= \frac{V_m I_m}{2} \cos \theta - \frac{V_m I_m}{2} (\cos 2\omega t \cos \theta \pm \sin 2\omega t \sin \theta) \\
&= \frac{V_m I_m}{2} \cos \theta - \frac{V_m I_m}{2} \cos (2\omega t \pm \theta) \\
&= \frac{V_m I_m}{2} \left[\cos \theta + \sin \left(2\omega t \pm \theta - \frac{\pi}{2} \right) \right] \qquad (a\text{–}11)
\end{aligned}$$

where the current is the reference and the voltage is considered to be in phase displacement with the current. This is a wave with a frequency double that of the system and is merely the history of the power change with time (from instant to instant), not a sine curve that may be referred to a vector as in the instance of current and voltage. In special circuit work, these power curves are observed and analyzed.

A general expression for the average power of a complete period is given by

$$P = \frac{1}{T} \int_0^T vi \, dt$$

where T is the period and t is the time.

To make all the proportionality factors unity and the units the same for both a-c and d-c systems, these units have been so defined that the average power in either system will produce the same heating effect in a given pure resistance for a fixed period of time.

4. Power for Sinusoidal Waves and Direct Current. In the d-c system, the values of V and I are the same for every instant; therefore, by inspection, it may be seen that the general equation will take the form

$$P = \frac{1}{T}\int_0^T vi\,dt = VI$$

since both V and I are constant during any complete period. Here, again, it is understood that steady state has been established and that the ripple is negligible.

For the sinusoidal wave the condition is somewhat different, for

$$P = \frac{1}{2\pi}\int_0^{2\pi} vi\,dt \qquad (b\text{–}11)$$

and for the general condition v and i are

$$i = I_m \sin \omega t$$

$$v = V_m \sin (\omega t \pm \theta)$$

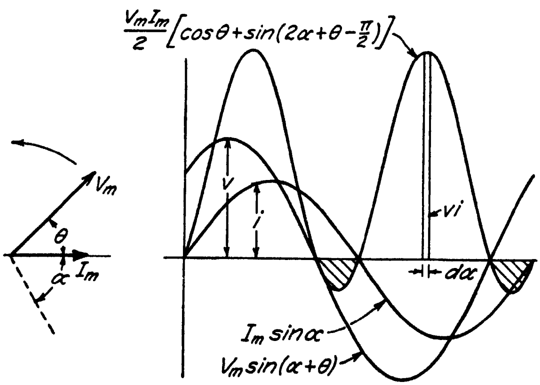


FIG. 1–11. The resultant power wave for a sinusoidal voltage and current, θ degrees out of phase.

which may be substituted into the above expression (b–11). To simplify the work of solving the integral, it is better to change the whole into

angular ($\alpha = \omega t$) rather than time intervals; consequently (Fig. 1-11) the integral will take the form

$$P = \frac{1}{2\pi}\int_0^{2\pi} vi\,d\alpha$$

$$i = I_m \sin\alpha$$

$$v = V_m \sin(\alpha \pm \theta)$$

$$P = \frac{1}{2\pi}\int_0^{2\pi} V_m I_m \sin(\alpha \pm \theta)\sin\alpha\,d\alpha$$

$$P = \frac{V_m I_m}{2\pi}\int_0^{2\pi} (\sin\alpha\cos\theta \pm \cos\alpha\sin\theta)\sin\alpha\,d\alpha$$

$$= \frac{V_m I_m}{2\pi}\int_0^{2\pi} (\sin^2\alpha\cos\theta \pm \sin\alpha\cos\alpha\sin\theta)\,d\alpha$$

$$= \frac{V_m I_m}{2\pi}\left[\cos\theta\int_0^{2\pi}\sin^2\alpha\,d\alpha \pm \sin\theta\int_0^{2\pi}\sin\alpha\cos\alpha\,d\alpha\right]$$

The last term will be zero, when integrated from 0 to 2π.

$$P = \frac{V_m I_m}{2\pi}\cos\theta\int_0^{2\pi}\sin^2\alpha\,d\alpha \pm 0$$

$$= \frac{V_m I_m}{2\pi}\cos\theta\left[\frac{\alpha}{2} - \frac{\sin 2\alpha}{4}\right]_0^{2\pi}$$

$$= \frac{V_m I_m\cos\theta}{2\pi}\times\pi = \frac{V_m I_m}{2}\cos\theta$$

$$= \frac{V_m}{\sqrt{2}}\times\frac{I_m}{\sqrt{2}}\times\cos\theta = VI\cos\theta \qquad (c\text{–}11)$$

where, by definition, the effective values of V and I are measured in units of the d-c voltage and current.

This gives the expression for the power in an alternating system, which is the product of the effective current, the effective voltage, and the power factor. Cosine θ is called the power factor because it represents the percentage of VI which is effective in producing power in the a-c system. VI is called the volt-amperes of the system or the apparent

power, whereas $VI \cos \theta$ is the true power (average, active, or real power). Another expression for the power factor is

$$\text{Power factor} = \cos \theta = \frac{\text{active power}}{\text{apparent power}} = \frac{P}{VI}$$

Example a. An impedance of 10 ohms is placed across a 100-volt a-c system. If a wattmeter connected into the system measures 800 watts, what will be the resistance, reactance, and power factor of the system?

$$I = \tfrac{100}{10} = 10 \text{ amp}$$

$$\cos \theta = \frac{800}{100 \times 10} = 0.8 \text{ power factor}$$

$$R = \frac{800}{(10)^2} = 8 \text{ ohms resistance}$$

$$X = \sqrt{10^2 - 8^2} \quad \text{or} \quad X = Z \sin \theta$$

$$X = \sqrt{36} = 6 \text{ ohms reactance}$$

5. Power Expressions. There are numerous expressions for power, and the form of the individual expression depends upon the information available for determining the power. Familiarity with the various forms facilitates rapid calculation, regardless of how the problem is presented.

In the d-c system, the combination of expressions for Ohm's Law and power gives

$$\text{Ohm's Law} \quad V = IR$$

$$\text{Power} \quad P = VI$$

$$P = (IR)I = I^2R$$

$$P = V\left(\frac{V}{R}\right) = \frac{V^2}{R}$$

$$P = V^2 \frac{1}{R}$$

$$P = V^2 g$$

Equation $P = VI$ is the most frequently used in the determination of power in a source or sink; equation $P = V^2/R$ is used for determination of the power consumed in meter coils; and equation $P = V^2g$ often obviates the necessity of reconverting conductances into resistances.

Considering a-c power—if either V or I is used as a reference, the expression may be converted into simpler forms. Figures 2–11 and 3–11 are used in conjunction with the general expression for power.

$$P = VI \cos \theta$$

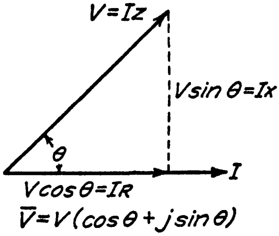


FIG. 2–11. Voltage triangle showing the relationship between the resistance, reactance, and impedance voltage drop.

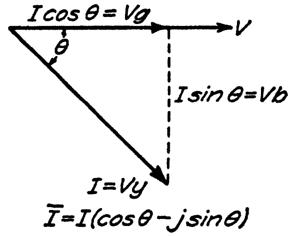


FIG. 3–11. A current triangle showing the relationship between the conductance, susceptance, and admittance current.

From Ohm's Law for the a-c system,

$$V = IZ = I\sqrt{R^2 + (X_{\mathfrak{L}} - X_C)^2}$$

By drawing the vector diagram, Fig. 2–11, which represents the voltage relationships, it will be seen that when the power expression is written as

$$P = I(V \cos \theta)$$

the $V \cos \theta$ term is the IR drop of the system. Therefore,

$$P = I^2R \qquad (d\text{–}11)$$

which indicates that only the resistance causes a loss in the system. If, however, the power expression is written as

$$P = V(I \cos \theta)$$

and reference is made to the following expression:

$$I = VY = V\sqrt{g^2 + (b_C - b_{\mathfrak{L}})^2}$$

and to Fig. 3–11, the $I \cos \theta$ term being the current component caused by the conductance of the system. By substitution,

$$P = V^2 \frac{R}{Z^2}$$

$$= V^2 g \qquad (e\text{–}11)$$

Though the general forms for the determination of power in sources and sinks in the d-c and a-c systems are different, the resistance and conductance forms are the same, and the use of resistance and conductance does not differ in the two systems. In determining power in a system containing both resistance and reactance, the conductance for the a-c and d-c systems are

Alternating current:

$$g = \frac{R}{R^2 + X^2} = \frac{R}{Z^2}$$

Direct current:

$$g = \frac{1}{R} \text{ (since } X = 0)$$

6. Power in a Pure Resistance. When a system contains only a pure resistance, the expressions for v and i are

$$v = V_m \sin \omega t$$

$$i = I_m \sin \omega t$$

The angle between them is zero, and $\cos \theta$ is unity. By substituting these values in the general form (a–11), the power curve is

$$p = \frac{V_m I_m}{2}\left[1 + \sin\left(2\omega t - \frac{\pi}{2}\right)\right] \qquad (f\text{–}11)$$

The average power of a system containing a pure resistance is

$$P = VI \cos \theta = VI \times 1$$

$$= VI$$

Figure 4–11 shows curves for power, current, and voltage for a resistance circuit. From the expression (when $X = 0$),

$$V = IZ = IR$$

A substitution in the power expression gives

$$P = I^2 R$$

or

$$P = \frac{V^2}{R} \quad \text{(only when } X = 0)$$

When the power curve for this condition is examined, it will be found that the product of the instantaneous voltage and current ($p = vi$ is the instantaneous power) will always be positive; that the power curve

lies above the axis and has a frequency double that of the system. This power is all positive; that is, the full value of the current is active in producing useful power. In general, all the power that lies above the reference axis is considered positive power.

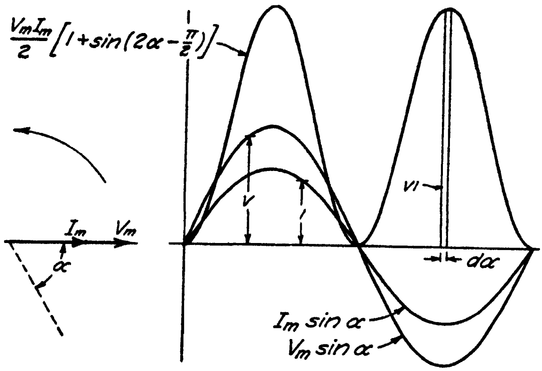


FIG. 4–11. The resultant power wave for a sinusoidal voltage and current in a system containing a pure resistance.

Example b. A pure resistance of 10 ohms is placed across a voltage of 100 volts. What will be the current flow if the current is alternating; if direct current? What will be the power under the two conditions? Determine the instantaneous power in both systems 0.01 sec after the switch is closed, assuming steady state with zero time occurring when the current and voltage pass through zero increasing toward a positive maximum. The frequency of the a-c system is 60 cycles.

Direct current:

$$I = \tfrac{100}{10} = 10 \text{ amp current}$$

$$P = 100 \times 10 = 1000 \text{ watts power}$$

Power is 1000 watts at any instant because v and i are always 100 volts and 10 amp.

Alternating current:

$$I = \frac{V}{Z} = \frac{100}{10} = 10 \text{ amp current}$$

$$P = VI\cos\theta = 100 \times 10 \times 1 = 1000 \text{ watts power}$$

$$v = V_m \sin\omega t$$

$$i = I_m \sin\omega t$$

$$\omega t = 2\pi \times 60 \times 0.01 = 3.77 \text{ radians}$$

$$3.77 \text{ radians} = 216^\circ \qquad \sin 216^\circ = -0.588$$

$$i = 10\sqrt{2} \times \sin 216^\circ = -8.31 \text{ amp}$$

$$v = 100\sqrt{2} \times \sin 216^\circ = -83.1 \text{ volts}$$

$$p = vi = 690.6 \text{ watts}$$

7. Effective Resistance. In the d-c system, all resistance is pure resistance and is the ohmic resistance; but, when a periodic current is passed through a resistance, especially when in the vicinity of iron, energy losses occur that are not chargeable to the resistance alone, but may be iron losses, skin effects, or both. These additional losses heat the system and dissipate energy in the same manner as does the resistance and, in an analysis, they are not easily differentiated from resistance losses. The normal treatment is to increase the magnitude of the ohmic resistance to compensate for the additional losses and to call the new resistance effective resistance. Effective resistance is measured as the quotient of the average rate of dissipation of electrical energy during a cycle of the periodic wave, divided by the square of the effective current. This is expressed by

$$R_{\text{eff}} = \frac{P}{I^2}$$

Example c. The resistance of a winding was determined with both direct current and alternating current. In the first instance, the voltage drop was 20 volts when 100 amp were flowing and, in the second instance, the current was 80 amp and the wattmeter read 1600 watts. What is the ohmic resistance and what is the effective resistance of the winding?

$$R = \frac{V}{I} = \frac{20}{100} = 0.2 \text{ ohm} \qquad \text{ohmic resistance}$$

$$R = \frac{P}{I^2} = \frac{1600}{(80)^2} = 0.25 \text{ ohm} \qquad \text{effective resistance}$$

Once the effective resistance has been determined for any operating condition, the value is used instead of the ohmic resistance, which is not correct. Computations are carried out in the same manner as those when pure resistance is used. Changes in current value and frequency will influence the effective resistance; therefore, care must be taken in using effective values of resistance. When the frequency is of the magnitude of radio frequencies, the ohmic resistance is a very small part of the total energy-consuming effective resistance.

8. Power in Pure Inductance. This problem, when treated in the same manner as that involving pure resistance, has the following expressions for v and i.

$$v = V_m \sin\left(\omega t + \frac{\pi}{2}\right)$$

$$i = I_m \sin \omega t$$

The power, current, and voltage curves are shown in Fig. 5–11 and, since the angle between the current and the voltage is 90°, the value of

$\cos\theta$ is zero. Substituting the value of $\cos\theta$ in the power expression, the average power is

$$P = VI\cos\theta = VI \times 0$$
$$= 0$$

Though the average power is zero, the instantaneous power is as significant as in a system of pure resistance. It will be noted from Fig. 5–11 that the power curve is symmetrical about the reference axis, having equal portions above and below.

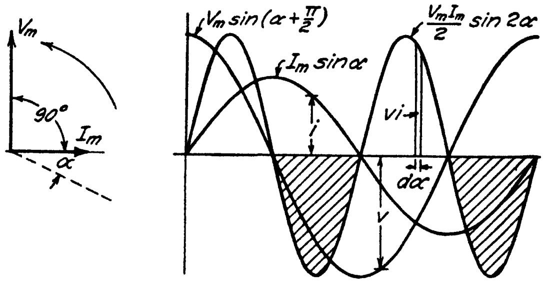


FIG. 5–11. The resultant power wave for a sinusoidal voltage and current in a system containing a pure inductance.

By substituting $\cos\theta$ equals zero and θ equals a positive angle of 90° in the general equation (a–11), the power curve will be expressed by

$$p = \frac{V_m I_m}{2} \sin 2\omega t \qquad (g\text{–}11)$$

This is a wave with a frequency double that of the system and similar to the curve for a resistance circuit. During the positive portion of the cycle, energy is being supplied by the source; during the negative portion, energy is returned by the collapse of the magnetic field in the inductance. Since the positive and negative areas are equal, there is no net transfer of energy from the source and the average power is zero.

In a resistance, the power flow is called active (true power). In the inductive system, the current is at quadrature with the voltage and, voltage being used as a reference, the current has only a sine component, the cosine component being zero. The product of voltage and current in the pure inductance is called reactive volt-amperes.

In developing the expressions $P = I^2R$ and $P = V^2g$, use was made of the in-phase components of V and I. The quadrature components

of V and I will be $V \sin \theta$ and $I \sin \theta$, respectively. The reactive volt-amperes may be written as

$$Q = I(V \sin \theta) = I(IX) = I^2X \qquad (h\text{–}11)$$

$$Q = V(I \sin \theta) = V(Vb) = V^2b \qquad (i\text{–}11)$$

In comparing the pure resistance and pure inductance systems, the following is true.

Pure Resistance (True Power)	Pure Inductance (Reactive Power)
$P = VI \cos \theta$	$Q = VI \sin \theta$
$Q = 0$	$P = 0$
$\cos \theta = 1$	$\sin \theta = 1$
$\sin \theta = 0$	$\cos \theta = 0$

The first contains no sine or reactive component, whereas the latter contains no cosine or active component of current.

Substituting the values for V and I in the power expression of the pure inductance system, it follows that, when

$$V = IX_{\mathfrak{L}} \qquad (j\text{–}11)$$

$$Q = I^2X_{\mathfrak{L}} \quad (Q\text{, reactive volt-amperes})$$

$$= \frac{V^2}{X_{\mathfrak{L}}} \quad (\text{only when } R \text{ is not present})$$

The energy stored in a magnetic field, though often used to do work, must frequently be dissipated to protect switching equipment. The magnitude of this energy may be determined by integrating a curve, between definite limits, representing the energy delivered from the source to the inductance. The limits of integration cover a quarter-period, since the power is double frequency and is charged and discharged every half-period. Expressing the energy in mathematical form,

$$J = \int_0^{T/4} VI \sin 2\omega t \, dt = \int_0^{\pi/2} VI \sin 2\alpha \, d\alpha$$

$$J = VI\left(-\frac{1}{2\omega} \cos 2\omega t\right)_0^{T/4} = \frac{VI}{\omega}\left(-\frac{1}{2} \cos 2\alpha\right)_0^{\pi/2}$$

$$J = \frac{VI}{\omega}$$

but, from equation (j–11),

$$V = IX_{\mathfrak{L}} = I\omega\mathfrak{L}$$

and

$$J = I\omega\mathfrak{L} \times \frac{I}{\omega} = I^2\mathfrak{L}$$

The effective value for current may be written

$$I = \frac{I_m}{\sqrt{2}}$$

$$J = \tfrac{1}{2}I_m{}^2\mathfrak{L}$$

as previously developed in Chapter 4. It will be noted that the energy stored depends upon the maximum current in the a-c circuit but, if the system is supplied with a direct current, the maximum current will be the steady-state current after the system has passed through its transient period. No energy is required to maintain the magnetic field once it is established, and the absorption of energy in an electromagnet is caused by the copper losses in the winding.

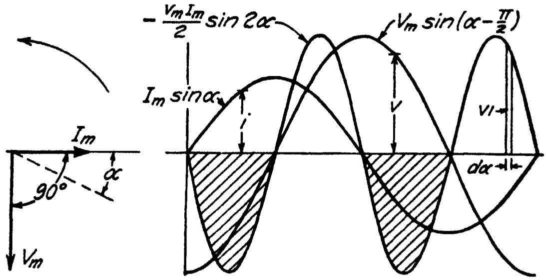


FIG. 6–11. The resultant power wave for a sinusoidal voltage and current in a system containing a pure capacitance.

9. Power in Pure Capacitance. When a sinusoidal current flows through a capacity, the instantaneous current and voltage are

$$i = I_m \sin \omega t$$

$$v = V_m \sin\left(\omega t - \frac{\pi}{2}\right)$$

By substituting $\cos\theta$ equal to zero and θ equal to $-90°$ in the general equation (a–11), the curve for the instantaneous power will be represented by

$$p = -\frac{V_mI_m}{2}\sin 2\omega t \qquad (k\text{–}11)$$

This is a wave having a frequency double that of the system (Fig. 6–11), but this power wave is 180° out of phase with a power wave for an in-

ductance system. The negative sign does not indicate a negative power but, rather, the relationship of the power to the original reference of current.

If the power curve is integrated to determine the average power, the loops, being symmetrical about the axis, will have an average of zero. The angle between the current and voltage is 90° as in a system of pure inductance; therefore, $\cos\theta$ is zero, which, when substituted into the expression for power, gives

$$\begin{aligned} P &= VI\cos\theta = VI \times 0 \\ &= 0 \end{aligned}$$

This system requires reactive volt-amperes as in the instance of inductance. The expression for the reactive volt-amperes is

$$\begin{aligned} Q &= VI\sin\theta \qquad \sin\theta = 1 \\ &= VI \end{aligned}$$

Substituting values for V and I,

$$\begin{aligned} V &= IX_C \\ Q &= I^2X_C \\ &= \frac{V^2}{X_C} \quad \text{(only when } R \text{ is not present)} \end{aligned}$$

Comparing this with the inductance condition, the energy in the capacity is stored and returned to the system by the dielectric field twice during a cycle. Integration of the expression

$$J = \int_0^{T/4} -VI\sin 2\omega t\,dt = \int_0^{\pi/2} VI\sin 2\alpha\,d\alpha$$

gives

$$J = -\frac{VI}{\omega} \tag{l–11}$$

similar to a system composed of pure inductance. However, in a pure capacity,

$$V = IX_C = \frac{I}{\omega C}$$

Therefore, from (l–11), neglecting the sign,

$$J = \frac{I}{\omega C} \times \frac{I}{\omega} = \frac{I^2}{\omega^2 C}$$

or

$$J = \frac{V}{\omega} \times V\omega C = V^2C$$

which, from consideration of the maximum values for the a-c system, will become

$$J = \tfrac{1}{2}V_m{}^2C$$

as previously developed in Chapter 5. If a condenser is placed on a d-c system, the energy stored when steady state has been reached depends upon the voltage of the system and the capacity of the condenser.

When the two instantaneous powers in an inductance and a capacitance are compared, there is (in the general expression) only a difference in sign:

$$p = vi$$

$$p \text{ in inductance } = VI \sin 2\omega t$$

$$p \text{ in capacitance } = -VI \sin 2\omega t$$

The opposite sign shows that, if the two were placed in the same system, they would transfer energy from one to the other without disturbing the power source. This condition is approached in a resonant system.

The average power consumed by *inductance* and *capacity* is *zero* and only *resistance consumes power* during the period of operation; however, the instantaneous power conditions existing in an inductance and capacitance are of commercial value. The energy stored in the induction coil of a closed circuit ignition system and the use of condenser charges for timing are examples of major commercial importance.

10. Active Power and Reactive Volt-Amperes. It has been the practice to consider only true power (called active power). The existence of reactive volt-amperes (called reactive power) and their influence on the economics of generation and distribution have been recognized, and the cost of larger generating units and distribution systems charged to overhead and shared by all consumers. Recently, however, there has been a tendency to place the cost where it belongs by charging the low power-factor consumer on a kilovolt-ampere (kv-a) base instead of the kilowatt (kw) base.

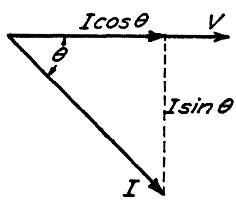


FIG. 7-11. Vector diagram showing the active and reactive component of current in power considerations.

Considering the vector diagram, Fig. 7-11, in which a system containing both resistance and inductance draws I amperes from the line, the current I flows through the generator and the system, producing losses measured by

$$P_L = I^2R$$

where R is the total resistance of the generator and transmission circuit. All the current flowing produces losses, and only the $I \cos \theta$ does useful work, for

$$P = VI \cos \theta$$

The other component of current does not work, being merely stored energy in the inductance for a part of a cycle, and is then returned to the source. The component $I \sin \theta$ is called the quadrature component while $I \cos \theta$ is the in-phase or energy component of current. The voltage may combine with either of the two components forming

$$P = VI \cos \theta \quad \text{(watts)}$$

$$Q = VI \sin \theta \quad \text{(vars)}$$

These two expressions are the active power and the reactive volt-amperes, respectively, as has been explained.

It is evident from the foregoing that the consumer should pay for all the service rendered and that extra overhead, on account of the large conductors, oversize transformers, and generators, which the company must supply, does not depend alone upon the power used but also upon how efficiently the current is used. When the power factor is unity and the quadrature component of current is zero, the system is delivering energy with the greatest current efficiency.

The consideration of power has several special units and terms, application of which depends upon the type of consumption. In direct current, the power is measured by

$$P = VI$$

(where V is the voltage and I is the current) because the power factor is unity. In this type of system the current and the voltage are constant (comparing alternating and direct current through complex expressions) and in phase with each other. The unit by which power is measured is the *watt*, but in some considerations this is too small and the *kilowatt* (1000 watts) becomes the normal unit.

In the a-c system, the product of the volts and amperes VI is called the apparent power and is spoken of as the *volt-amperes* of the system. The 10^{-3} multiple of volt-amperes is a *kilovolt-ampere*. If the volt-ampere is multiplied by the cos θ (the power factor), the expression is

$$P = VI \cos \theta \quad \text{(watts)}$$

normally considered in power sources and measured in watts or kilowatts. However, if the sine function of the power-factor angle is used, the expression is

$$Q = VI \sin \theta \quad \text{(vars)}$$

which is measured in *vars* or reactive volt-amperes, the *kilovar* being used to measure large quantities of reactive power. The kilovar equals vars $\times$ 10^{-3}.

Charging for electrical service on a volt-ampere second base, or kilovolt-ampere hour base, places the cost of poor power-factor load where it belongs; and wherever there are large power loads this is becoming the practice.

11. Joule's Law (1841). This law states that the heat produced by an electrical current in a circuit of constant resistance is directly proportional to the square of the current and the time. In mathematical form,

$$J = I^2Rt \qquad (m\text{–}11)$$

If this is combined with the expression for Ohm's Law, the energy is

$$J = VIt$$

This holds for alternating current if the opposition is a pure resistance and all other effects are eliminated. The foregoing expressions are correct when average values are taken but, for periodic conditions, the form

$$J = \int vi\, dt$$

which is the general expression for calculating energy.

The basic unit of heat is the *joule* or the *watt-second,* and is also the basic unit of energy in the electrical system. Equation (m–11) may be changed to represent gram-calories by dividing by the factor 4.186 (the watt-seconds or joules in the gram-calorie). Making this conversion,

$$J = \frac{1}{4.186} I^2Rt = 0.239I^2Rt \text{ calories}$$

where I is in amperes, t is in seconds, and R is in ohms. This equation may be used in calculating the heating effect of electricity, but there are corrections to be made for radiation and conduction which must be known in each specific problem. Since electrical power and energy units are related to heat units, it is a simple matter to make conversions from mechanical units to electrical units; this is often necessary.

12. Conversion Factors. The following common units of power and energy are useful in converting from mechanical to electrical units and *vice versa*:

1 erg	= 1 dyne cm	
1 joule	= 10^7 ergs	= 1 watt-sec
1 ft-lb	= 1.356 joules	= 1.356 watt-sec
1 whr	= 3.6×10^3 joules	= 2.655×10^3 ft-lb
1 kwhr	= 3.6×10^6 joules	= 2.655×10^6 ft-lb
1 hp-hr	= 2.684×10^6 joules	= 0.7457 kwhr
1 gm-cal	= 4.186 joules	= 3.088 ft-lb
1 Btu	= 1055 joules	= 778.1 ft-lb
1 watt	= 1 joule per second	
1 kw	= 1.34 hp	= 1000 watts
1 hp	= 746 watts	= 0.746 kw

In the conversion from kilowatts to horsepower it is sufficiently accurate, and often simpler, to use

$$1 \text{ hp} = \tfrac{3}{4} \text{ kw (approximately)}$$

$$1 \text{ kw} = \tfrac{4}{3} \text{ hp (approximately)}$$

13. Metering Losses and Measurement of Single Phase and D-C Power. Figure 8–11 shows an ammeter and voltmeter connected in a circuit to measure either single phase (if power factor is known) or d-c power.

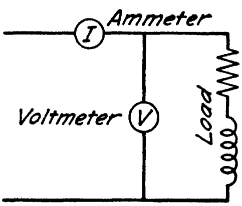


FIG. 8–11. Meter connections for measuring d-c power.

When instruments are used for measuring various values in electrical systems, the losses in the instruments are small and, usually, can be neglected. However, it should be remembered that the introduction of an instrument into a system is equivalent to placing an impedance in series or parallel with the system load and that energy will be dissipated in proportion to the resistance of the instrument and the amount of current in the instrument circuit. In Fig. 8–11, if the voltmeter loss is large enough to be an appreciable part of the load being measured, the power should be corrected by subtracting the voltmeter loss, and the actual power will be

$$P = VI \cos \theta - \frac{V^2}{R}$$

where V and I are the volts and amperes indicated by the meters, and R is the resistance of the voltmeter. The foregoing connection (Fig. 8–11)

is more satisfactory than that in which the voltmeter measures the loss in the ammeter, because, as the potential is usually constant, one correction will serve for the whole set of observations.

Figure 9–11 shows the usual method of connecting a wattmeter for the determination of power in a single-phase load. Here, again, the correction for the power loss in the instrument is made by subtracting the losses in the potential coil from the wattmeter reading. Since this is a dynamometer type of instrument, the power indicated is the true power and not the apparent power. The instrument should always be so connected that the ± binding post of the potential coil is on the same side of the circuit as the current coil. This is necessary because the electrostatic attraction between the current and potential coils might otherwise introduce an error in the reading.

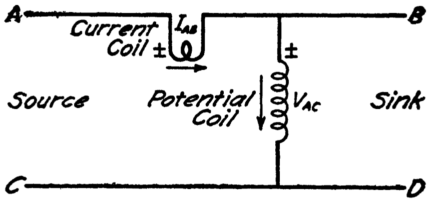


FIG. 9–11. Proper connection of wattmeter for measuring power (the conventional representation of the wattmeter to be used in the text).

A wattmeter may be used in measuring the power in a d-c system, but it is necessary to take an average of two readings with the meter connected directly and reversed (both current and potential coil reversed). This is to correct for any effect which the forming of definite polarities in the magnetic shielding might have on the readings.

14. Calculation of Power from Complex Expressions for the Current and Voltage in the System. Frequently it is desirable to determine the power in a system when only the complex values are known. The power will be, *not the product of the complex quantities*, but the algebraic sum of the product of the real components and the product of imaginary components. For example,

$$\bar{V} = v + jv' \quad \text{(in symbolic components)}$$

$$\bar{I} = i + ji' \quad \text{(in symbolic components)}$$

$$P = vi + v'i'$$

This statement may be easily proved. Consider Fig. 10–11 and the general equation for power, which is

$$P = VI \cos \theta$$

Substituting the angle $(\beta \mp \alpha)$ for the power-factor angle θ and expanding the difference of two angles gives

$$P = VI \cos (\beta \mp \alpha)$$

$$= VI (\cos \beta \cos \alpha \pm \sin \beta \sin \alpha)$$

$$= VI \left(\frac{v}{V} \times \frac{i}{I} + \frac{v'}{V} \times \frac{i'}{I} \right) \quad \text{(When } \alpha \text{ is positive, } \theta = \beta - \alpha.\text{)}$$

$$= VI \left(\frac{v}{V} \times \frac{i}{I} - \frac{v'}{V} \times - \frac{i'}{I} \right) \quad \text{(When } \alpha \text{ is negative, } \theta = \beta + \alpha.\text{)}$$

$$P = \frac{VI}{VI} (vi + v'i')$$

$$= (vi + v'i')$$

Referring to Fig. 10–11, these represent the in-phase and quadrature components of V and I where the power is the product of the in-phase components of V and I plus the product of the quadrature components of V and I.

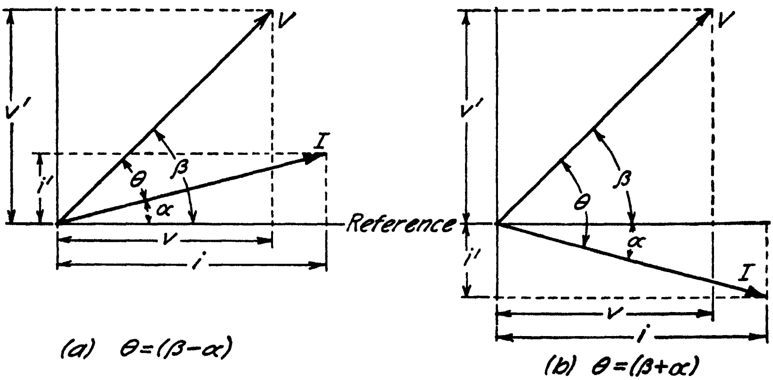


FIG. 10–11. The in-phase and quadrature components of current and voltage, used in determining the power by means of symbolic expressions.

It is essential, when using this method of power determination, that the proper vector current and voltage be used, for the signs of the two products must be correct to avoid error in the computed power. *Attention must be given to the algebraic sign of each component,* which may be either positive or negative.

Figure 11–11 shows a source and a sink with a wattmeter connected to indicate the power. If the potential coil is connected to one side of the current coil so that the flow through both the potential and current coils is simultaneously in the same direction, the potential and current

subscripts are read in the direction of the arrows and the resultant computation of power will be correct. The values for voltage and current to be used in Fig. 10–11 will be

Power from $\bar{I}_{AB}$ and $\bar{V}_{AC}$, expressed symbolically by

$$P = \overline{\bar{V}_{AC} \,|\, \bar{I}_{AB}}\ *$$

indicates that the product of the real terms and the product of the imaginary terms are to be added algebraically.

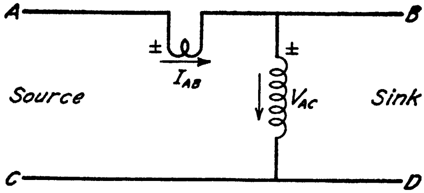


FIG. 11–11. Subscript method of reading the proper voltage and current (in symbolic form), to be used in the determination of power.

Assuming the following expression for the complex current and voltage,

$$\bar{V}_{AC} = a + jb$$

$$\bar{I}_{AB} = c - jd$$

the power determined from the complex quantities will be

$$P = \overline{a + jb \,|\, c - jd} = ac - bd$$

To determine the reactive volt-amperes, namely,

$$Q = VI \sin \theta$$

from complex expressions, substitute $(\beta \mp \alpha)$ for θ, $\sin (\beta \mp \alpha) = \sin \beta \cos \alpha \mp \cos \beta \sin \alpha$ and, solving as was done previously, in the instance of true power, the resultant equations are

$$Q = VI\left(\frac{v'}{V} \times \frac{i}{I} - \frac{v}{V} \times \frac{i'}{I}\right)$$

$$Q = \frac{VI}{VI}(v'i - vi') = v'i - vi'$$

Use

$$Q = vi' - v'i \qquad (n\text{–}11)$$

* These two symbolic forms for active and reactive power from complex quantities are not standard; neither have they been adopted for use by the profession. They have been used to clarify the indication of these values.

$$P = \overline{A \,|\, B} \qquad Q = \underline{A \,|\, B}$$

to satisfy A.S.A. definition.† The voltage and current to be used from Fig. 11–11 will be $\bar{I}_{AB}$ and $\bar{V}_{AC}$, expressing reactive volt-amperes symbolically by

$$Q = \underline{\bar{V}_{AC} \mid \bar{I}_{AB}} \text{ † ‡}$$

indicating that the product is a cross product of the real and imaginary terms having the algebraic signs indicated by equation (n–11) so that *leading current will give positive † reactive volt-amperes.*

Substituting in this equation the values given for $\bar{I}_{AB}$ and $\bar{V}_{AC}$, the values for the reactive volt-amperes will be

$$Q = \underline{a + jb \mid c - jd} = (a \times -d) - (b \times c)$$
$$Q = -ad - bc$$

determined from the complex expressions for the voltage and current.

Volt-amperes, like impedance, may be expressed in complex form, even though scalar in nature. From the power expression,

$$P = VI \cos \theta$$
$$VI = \frac{P}{\cos \theta}$$
$$VI = \frac{P}{\sqrt{1 - \sin^2 \theta}}$$
$$(VI)^2 = \frac{P^2}{1 - \sin^2 \theta}$$
$$(VI)^2 - (VI)^2 \sin^2 \theta = P^2$$
$$VI = \sqrt{P^2 + (VI)^2 \sin^2 \theta} = \sqrt{P^2 + (VI \sin \theta)^2}$$
$$VI = \sqrt{P^2 + Q^2}$$
$$\bar{P}_A = P \pm jQ$$
$$\cos \theta = \frac{P}{\sqrt{P^2 + Q^2}}$$

† The negative sign for inductive reactive power is used because in a power system the inductance requires excitation and the capacity furnishes excitation. As reported by Wagner and Evans in "Symmetrical Components" the preponderance of the published literature assigns a positive sign to the reactive power of inductance. However, the American Standards Definitions (05.21.165) assigns a positive sign when a capacitive circuit connects the point of entry to the reference point.

‡ These two symbolic forms for active and reactive power from complex quantities are not standard; neither have they been adopted for use by the profession. They have been used to clarify the indication of these values.

$$P = \overline{A \mid B} \qquad Q = \underline{A \mid B}$$

Expressed in words, the complex apparent power is equal to the sum of the true power plus the reactive volt-amperes added at 90°. Figure 12–11 shows the volt-ampere triangle and the angle θ (which is the power-factor angle). The tangent of the angle θ is

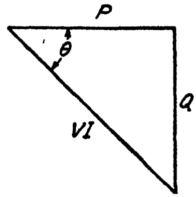


FIG. 12–11. A power triangle showing the relationship between the active power (P), the reactive power (Q), and volt-amperes (VI).

$$\tan \theta = \frac{Q}{P}$$

This form for expressing apparent power is useful in the study of reactive volt-amperes.

Example d. The voltage and current of a single-phase a-c system are $100 + j0$ and $30 - j10$, respectively. Determine the volt-amperes, the true power, and the reactive volt-amperes of the system, using the power factor and the various equations developed.

$$\bar{V} = 100 + j0$$

$$V = 100$$

$$\bar{I} = 30 - j10$$

$$I = 31.62$$

$$\cos \theta = \frac{30}{31.62} = 0.949 \quad \text{(from } I \text{ where } V \text{ is the reference)}$$

$$\sin \theta = \frac{-10}{31.62} = -0.316$$

$$P = 100 \times 31.62 \times 0.949 = 3000 \text{ watts true power}$$

$$VI = 100 \times 31.62 = 3162 \text{ volt-amperes apparent power}$$

$$Q = 100 \times 31.62 \times -0.316 = -999.2 \text{ vars reactive volt-amperes}$$

Example e. Solve example *d* by using the complex expressions for V and I. Check the results with those obtained in example *d*.

$$P = \overline{100 + j0 \,|\, 30 - j10}$$

$$= (30 \times 100) + (0 \times -10) = 3000 \text{ watts true power}$$

$$Q = \underline{100 + j0 \,|\, 30 - j10}$$

$$= (100 \times -10) - (0 \times 30) = -1000 \text{ vars reactive volt-amperes}$$

$$\bar{P}_A = P + jQ$$

$$P_A = \sqrt{3000^2 + 1000^2} = 3162 \text{ volt-amperes apparent power}$$

In the consideration of the functions of the angle θ, the voltage is the reference and $\sin\theta$ is negative, making reactive power negative with a lagging current.

15. Summary. The following are the various forms for power:

Active Power (P in watts)	Reactive Power (Q in vars)	Apparent Power (P_A in volt-amperes)	When Used (given)
$VI\cos\theta$	$VI\sin\theta$	VI	V, I, pf
I^2R	I^2X	I^2Z	R, X, I, or Z, pf, I
V^2g	V^2b	V^2Y	g, b, V, or Y, pf, V
$vi + v'i'$	$vi' - v'i$	$P \pm jQ$	$\bar{V}, \bar{I}$

In any system (series, parallel, series-parallel, or network) the following is true for the power dissipated in all the sinks.

$$\begin{aligned}
&\text{Total power } (P) &&= \Sigma(P_1 + P_2 + P_3 + \cdots P_n) \text{ (numerical)}\\
&\text{Total reactive power } (Q) &&= \Sigma(Q_1 + Q_2 + Q_3 + \cdots Q_n) \text{ (algebraic)}\\
&\text{Total apparent power } (P_A) &&= \Sigma(\bar{P}_{A_1} + \bar{P}_{A_2} + \bar{P}_{A_3} \cdots \bar{P}_{A_n}) \text{ (symbolic)}\\
& &&= \Sigma P + j\Sigma Q
\end{aligned}$$

The instantaneous power curve is expressed by

$$\text{General} \qquad p = \frac{V_m I_m}{2}\left[\cos\theta + \sin\left(2\omega t \pm \theta - \frac{\pi}{2}\right)\right]$$

$$\text{Pure resistance} \qquad p = \frac{V_{m_R} I_m}{2}\left[1 + \sin\left(2\omega t - \frac{\pi}{2}\right)\right]$$

$$\text{Pure inductive reactance} \qquad p = \frac{V_{m_L} I_m}{2}\sin 2\omega t$$

$$\text{Pure capacitive reactance} \qquad p = -\frac{V_{m_C} I_m}{2}\sin 2\omega t$$

16. Power Locus Diagrams. In Art. 14, the power in a-c circuits was represented by non-rotating vectors. The symbolic treatment of power introduces definite restrictions upon the multiplication of vector currents and voltages. The summary in Art. 15 represents power in all its forms using the symbolic interpretation and the absolute value of current and voltage with circuit parameters.

Power loci, because of the special restrictions placed upon them, are limited to definite regions of operation, which in mathematical interpretation are called limits or boundaries. By fixing the voltage at the sending end (a condition desirable in the normal constant potential circuit) the apparent power locus has limits set by the square of the voltage and by the admittance expressed in a locus form. The loci of the parameters, as shown under parallel circuits, may have either con-

stant absolute values with variable power factors or fixed power factors with variable values for the admittance. The apparent power is expressed by

$$P_A = V^2Y$$

If a constant current system is considered, the apparent power depends upon the square of the current and the loci of the impedance. The apparent power will be

$$P_A = I^2Z$$

The impedance loci may be obtained by varying the power factor and keeping the absolute impedance constant or by keeping the power factor constant and varying the impedance.

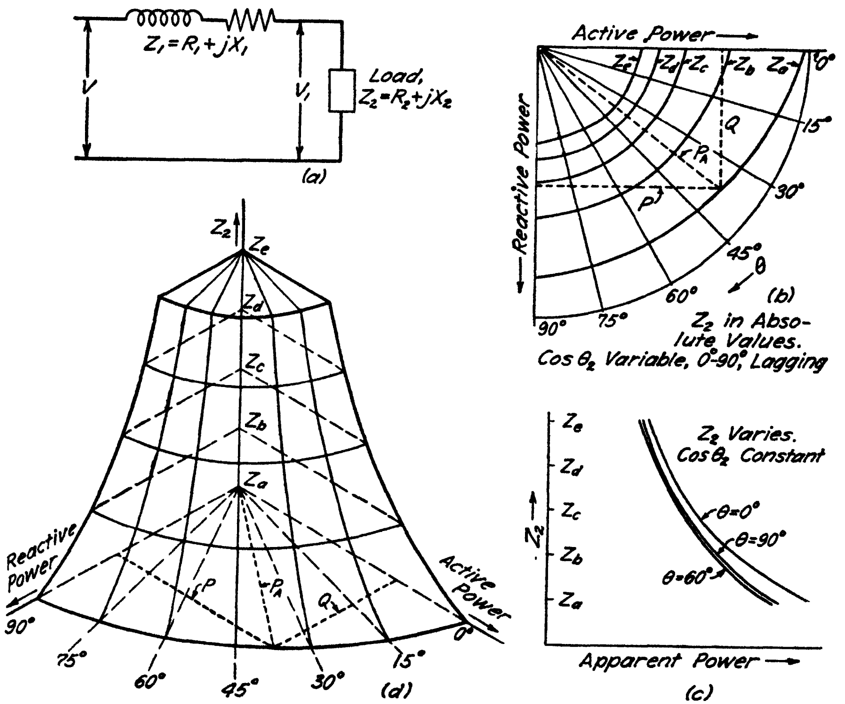


FIG. 13-11. Power circuit and the locus diagrams when the voltage (V), the impedance (Z_1), and the power factor (θ_1) are constant.

Since the active and reactive power have an interdependency caused by the relationship of the circuit parameters to the values of V and I, the problem of power loci is complex.

As a specific example, consider the circuit shown in Fig. 13-11a which represents a transmission line of fixed impedance Z_1 and a con-

stant sending end voltage V. The load Z_2 is an impedance which may vary, between reasonable limits, in both scalar value and power factor. The loci of the apparent power are encountered in the study of the performance of the usual line supplying a load.

The active, reactive, and apparent power for this system may be expressed in the absolute values of the system voltage, current, parameters, and power factors. To develop a general solution determine the values of

$$\bar{Z}_1 = R_1 + jX_1 \qquad \cos\theta_1 = \frac{R_1}{Z_1}$$

$$\bar{Z}_2 = R_2 + jX_2 = \frac{1}{\bar{Y}_2} \qquad \cos\theta_2 = \frac{R_2}{Z_2}$$

and the value of the system current

$$I = \frac{V}{\sqrt{(R_1 + R_2)^2 + (X_1 + X_2)^2}} \tag{o–11}$$

Substitute this value of current in the general expressions for the active, reactive, and apparent power and in the specific expressions for these in terms of the parameters of the circuits; thus the sending end voltage may be obtained.

Substituting equation (o–11) into the expression for active power,

$$P = I^2R_2$$

$$P = \frac{V^2R_2}{(R_1 + R_2)^2 + (X_1 + X_2)^2}$$

$$= \frac{V^2Z_2\cos\theta_2}{(R_1 + Z_2\cos\theta_2)^2 + (X_1 + Z_2\sin\theta_2)^2}$$

$$= \frac{V^2Z_2\cos\theta_2}{(R_1^2 + X_1^2) + Z_2^2(\cos^2\theta_2 + \sin^2\theta_2) + 2Z_2(R_1\cos\theta_2 + X_1\sin\theta_2)}$$

$$= \frac{V^2Z_2\cos\theta_2}{Z_1^2 + Z_2^2 + 2Z_1Z_2\left[\dfrac{R_1}{Z_1}\cos\theta_2 + \dfrac{X_1}{Z_1}\sin\theta_2\right]}$$

$$= \frac{V^2Z_2\cos\theta_2}{Z_1^2 + Z_2^2 + 2Z_1Z_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)}$$

$$= \frac{V^2Z_2\cos\theta_2}{Z_1^2 + Z_2^2 + 2Z_1Z_2\cos(\theta_1 - \theta_2)} \tag{p–11}$$

By a similar development, the reactive power will be

$$Q = I^2X_2$$
$$= \frac{V^2Z_2 \sin \theta_2}{Z_1^2 + Z_2^2 + 2Z_1Z_2 \cos(\theta_1 - \theta_2)} \qquad (q\text{–}11)$$

and the apparent power will be

$$P_A = I^2Z_2$$
$$= \frac{V^2Z_2}{Z_1^2 + Z_2^2 + 2Z_1Z_2 \cos(\theta_1 - \theta_2)} \qquad (r\text{–}11)$$

These three general equations, (p–11), (q–11), and (r–11), may be used in developing various loci for the load on the system. To develop the specific example of apparent power loci it will be assumed that V (sending end voltage) and Z_1 (line impedance) remain constant while the load changes. Figure 13–11b shows the loci when the absolute impedance Z_2 remains constant and the power factor is varied. Figure 13–11c shows the loci when the angle θ_2 is kept constant and the absolute value of the impedance is varied. Figure 13–11d shows these relationships in three dimensions and gives a form of locus diagrams which frequently appears in model form. These locus diagrams, representative of many mechanical devices, are published to save time in making computations. The construction of these diagrams and models requires time and application, but they are justified because a large number of calculations would otherwise have to be made. As in the demonstration problem, these loci are usually based upon some relatively simple development.

Locus diagrams, as shown in Figs. 13–11b and 13–11c, which are restricted to one plane, are frequently used in the study of machines and circuits.

PROBLEMS

1–11. An impedance draws 1200 watts from a 120-volt, $I = 10$ amp. at 60 cycles; however, when the frequency is changed to 30 cycles and the voltage is kept constant, the impedance draws 768 watts. Determine (a) the complex expression for the impedance at 30 cycles and (b) the power factor.

2–11. On a 100-volt, 60-cycle system an impedance draws 600 watts at a 60 per cent leading power-factor current. Determine (a) the current, (b) the circuit parameters, (c) the reactive power.

3–11. Determine both the a-c and d-c (a) power, (b) vars, and (c) volt-amperes, using the current, voltage, and power factor, for a 100-volt system and an impedance $Z = 4 + j3$.

4–11. Determine the values required in Prob. 3, using the impedance, power factor, and current.

5–11. Determine the values required in Prob. 3, using the admittance, power factor, and voltage.

6–11. Given the voltage $\bar{V}_0 = -50 + j86.6$ and the current $\bar{I}_0 = 7.07 + j7.07$ for an electrical system. Using these values determine (*a*) the active, (*b*) the reactive, and (*c*) the apparent power from the complex quantities.

7–11. Two impedances, $\bar{Z}_1 = 10 + j0$ and $\bar{Z}_2 = 8 + j6$, are connected in parallel across a 1-phase, 100-volt supply. Determine the power by all the methods.

8–11. Determine the values required in Prob. 3, using the complex expressions of the voltages and currents.

9–11. Three parameters R, Z, and X_C are connected in series across a 1-phase, 100-volt system. When 4 amp are flowing in the system the voltage drops across the three loads are 80, 60, and 90 volts respectively. What is the power taken by the impedance?

10–11. An impedance, and a wattmeter for measuring the power, are placed on a 1-phase, 100-volt line. The wattmeter has a voltage coil (consider pure resistance) of 6000 ohms and the voltage drop in the current coil is neglected. A variable inductance is placed in series with the wattmeter potential coil and varied until the wattmeter indicates zero; this requires 8000 ohms of inductive reactance. Determine (*a*) the complex expression for the impedance and (*b*) the power drawn if the current flow is 10 amp.

11–11. A load, $\bar{Z} = 13 + j19$, is supplied from a voltage source, $\bar{V}_0 = 150 + j200$, through two lines with impedances $\bar{Z} = 1.0 + j0.5$ each. Determine the (*a*) active, (*b*) reactive, and (*c*) apparent power of the load.

12–11. On a 1-phase 200-volt system 4 amp are supplied to the load which consists of R, X_C and Z_L in series with voltage drops of 170, 160, and 220 volts respectively. Using the circuit parameters, voltage, and current determine (*a*) the Z_L, without the use of trigonometry, (*b*) the total power, and (*c*) the power through the impedance.

13–11. Three impedances, $\bar{Z}_{BC} = 5 + j0$, $\bar{Z}_{CD} = 3 + j4$, and $\bar{Z}_{DA} = 4 - j3$, are connected in series across $\bar{V}_{AB} = 100 + j0$. A wattmeter potential coil is connected across the impedance Z_{CD}. Using complex quantities determine the reading of the wattmeter.

14–11. Compare the volt-amperes supplied the impedances in the two following arrangements:

(1) $\bar{Z}_1 = 7.07 - j7.07$ in parallel with $\bar{Z}_2 = 6 + j8$

(2) Another impedance, $\bar{Z}_3 = 12 - j16$ placed in parallel with Z_1 and Z_2

15–11. A capacitive impedance of 10 ohms requires 800 watts from a 100-volt system. Determine the complex expression for the impedance.

CHAPTER 12

POLYPHASE CIRCUITS

The use of single-phase power is confined largely to lighting, small motors, and household equipment. For this class of equipment, the variations in instantaneous power demand and the electrical transients caused by switching are not of prime importance.

The polyphase system consists of several single-phase systems so connected and displaced in time phase that the total load on all phases places a more constant load on the generating source and the prime mover. The analogy between a multicylinder engine and a single-cylinder engine is often used. Both the single-cylinder engine and the multicylinder engine may deliver the same average power output, but the output of the multicylinder engine is more uniform. In the polyphase system, as in the multicylinder engine, the timing or phase displacement is always considered important. Although the instantaneous power from any one phase of a polyphase system may be negative in value part of the time, the total power from all phases is constant for balanced conditions. This characteristic makes the polyphase system more desirable for transferring large amounts of energy.

1. Phase Rotation and Phase Sequence. Two terms used constantly in polyphase discussions are "phase rotation" and "phase sequence."

Phase rotation is the direction in which the vectors are considered to be rotating. In trigonometry, the vector is rotated counterclockwise when it is used to produce the transcendental functions of the angle. For all polyphase discussions, the rotation of all vectors shall be considered counterclockwise unless otherwise stated.

Phase sequence is the order in which the vectors representing the voltages and currents are considered in time phase with respect to each other for a given phase rotation.

If the phase sequence is ABC for three vectors, A, B, and C, the order of the vectors rotating counterclockwise past the position of the observer would be vector A, vector B, and vector C. In writing the complex expressions for a polyphase voltage supply, the phase sequence and

phase rotation must be known. Figure 1–12 shows a three-phase voltage supply for two different phase sequences. The importance of phase rotation and phase sequence will become more apparent as the polyphase systems are discussed.

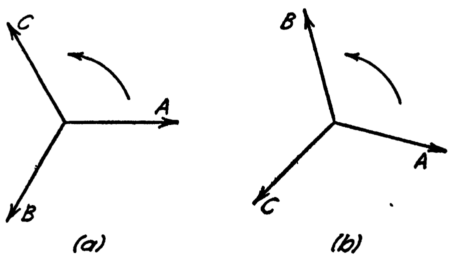


FIG. 1–12. Two different phase sequences: (a) ABC; (b) ACB.

2. Two-Phase Voltage Generation. In the single-phase alternator (Fig. 2–12) one winding consists of a coil for each pair of poles, and each coil side is placed symmetrically with respect to the poles. The instantaneous voltage $e_{A_1A_2}$ is equal at any instant to the sum of the voltages generated in the conductors, 1, 2, 3, and 4, at that instant, and will be a sine wave if the individual conductor voltages are sinusoidal. The voltage $e_{A_1A_2}$ can be represented as a vector, the length of which depends upon the addition of the vectors representing these conductor voltages. If the coil sides (conductors) are symmetrical with respect to the poles, the winding is a full pitch winding and the individual voltages produced in the conductors will be in time phase, and the vectors representing these voltages will be in the same straight line with the total voltage. If the coil sides do not occupy the same respective posi-

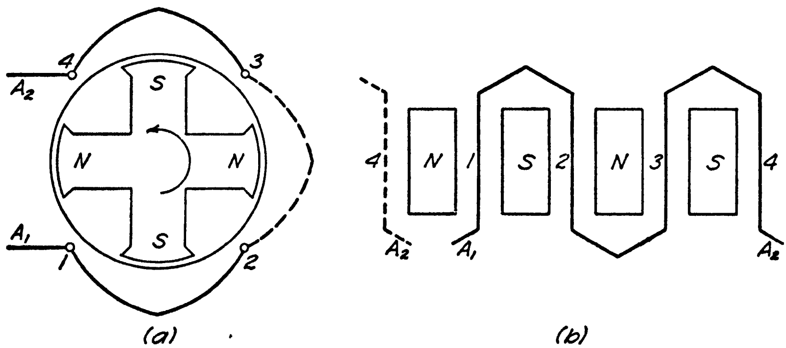


FIG. 2–12. (a) Coil displacement of a single-phase generator. (b) Developed winding showing the position of the coil sides.

tions in relationship to the poles, the winding is said to have a fractional pitch. For this condition the total voltage will be the vector sum of the voltages generated in the individual conductors and will be less than for a full pitch winding. In Fig. 3–12a, the total voltage and the conductor

voltages for a full pitch winding are shown. In Fig. 3–12*b*, the corresponding values for a fractional pitch winding are shown. Note the decrease in total voltage when the winding has a fractional pitch. It is not always advisable to construct a machine with full pitch windings,

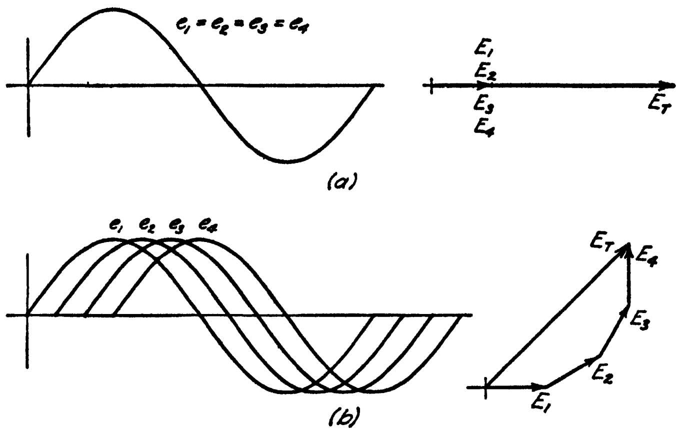


FIG. 3–12. (*a*) Four conductor voltages for a full pitch winding. (*b*) Conductor voltages of a fractional pitch winding.

because the slots may not be of the proper number or may be improperly spaced. Machines are also built with fractional pitch windings to reduce harmonics in the generated voltages caused by pulsations in the magnetic circuit flux and to simplify the connecting of the coil ends.

If a second single-phase generator, identical with the first, is connected to the same shaft (Fig. 4–12) and fastened mechanically to the

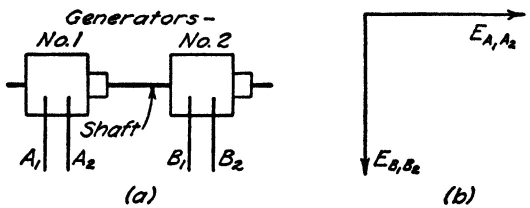


FIG. 4–12. (*a*) Two identical generators connected to the same shaft with generator No. 2 placed on the shaft to give a voltage 90 degrees behind No. 1. (*b*) Vector diagram of the two voltages.

shaft so that the generated voltage is displaced from the first generator voltage by 90 electrical degrees, there is available a two-phase supply of power from the two machines. If, however, a second set of coils is placed on one of the generators midway between the original windings, the same relationship exists between the generated voltages of these two

windings as exists between the two single-phase generators in Fig. 4–12. Figure 5–12 shows this construction, and the voltage $E_{B_1B_2}$ will be equal to the vector sum of the voltages generated in conductors 5, 6, 7, and 8.

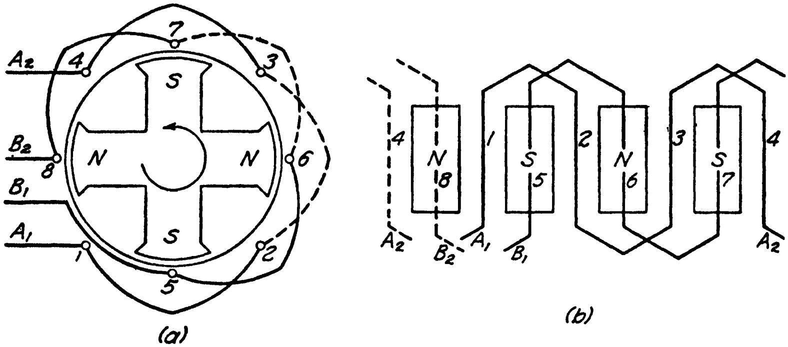


FIG. 5–12. (a) End connections of a two-phase generator. (b) Developed winding of coils showing 90-degree displacement of phases.

The two voltages will be equal in the two phases, and displaced in time phase by 90 electrical degrees because the two sets of coils are mechanically displaced to give this electrical time phase displacement. Because of the direction in which the field is rotated relative to the armature

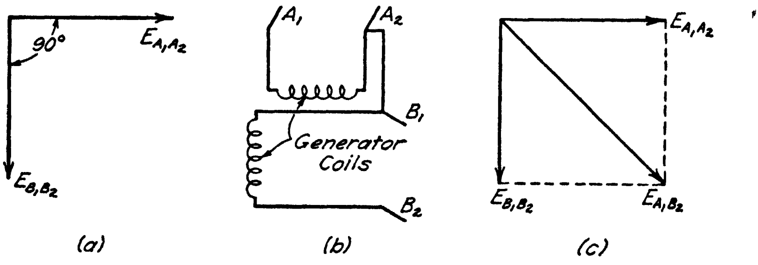


FIG. 6–12. (a) Phase relationship of $E_{A_1A_2}$ and $E_{B_1B_2}$. (b) Coil connections for two-phase, three-wire. (c) Vector diagram of voltages in part b.

windings, the voltage $E_{B_1B_2}$ lags the voltage $E_{A_1A_2}$ as indicated in Fig. 6–12a. If the two coils (Fig. 6–12b) are connected as indicated, the total voltage $E_{A_1B_2}$ (Fig. 6–12c) is obtained as

$$\bar{E}_{A_1B_2} = \bar{E}_{A_1A_2} + \bar{E}_{B_1B_2}$$

By connecting the coils together, as shown in Fig. 6–12b, it is possible to have two magnitudes of voltages and, if angular displacement of voltages is considered, three different values are available. The voltages are

$$\bar{E}_{A_1A_2}, \quad \bar{E}_{B_1B_2}, \quad \text{and} \quad \bar{E}_{A_1B_2}$$

In magnitudes, the values are

$$E_{A_1A_2} = E_{B_1B_2} \qquad E_{A_1B_2} = \sqrt{2}\, E_{A_1A_2}$$

3. Three-Phase Generation. If three identical generators are connected to the same shaft and each is keyed on the shaft so that the voltages are equal in magnitude and symmetrically displaced in time phase from each other, the combined output is a three-phase source. These voltages, when equally displaced in time phase, are 120 electrical degrees apart and independent of each other. Figures 7–12 and 8–12 show the three-phase source and the vector diagram. If three similar windings are placed on one alternator armature, 120 electrical degrees apart, the voltages generated in these windings will be displaced 120 electrical degrees. The machine is similar to the two-phase machine, except that the windings are displaced 120° in the three-phase generator instead of 90°. Figure 9–12 shows the coils and developed winding of a simple four-pole, three-phase generator.

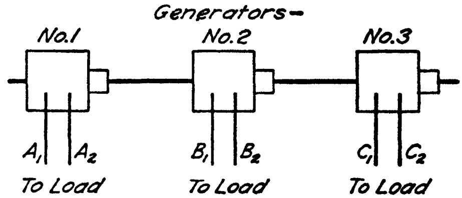


FIG. 7–12. Three single-phase generators on the same shaft supplying three separate loads. The generators are keyed on the shaft 120 electrical degrees apart.

The three voltages from a revolving armature machine can be made available by connecting the ends of each winding to a pair of slip rings. This would require six slip rings and give three independent sources of

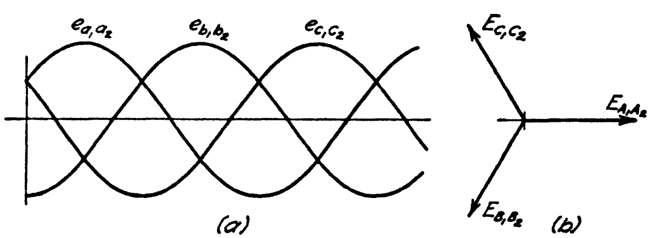


FIG. 8–12. (a) Voltage waves of a three-phase system. (b) Vector diagram of the voltages. $e_{a_1a_2} = E_m \sin \omega t$, $e_{b_1b_2} = E_m \sin (\omega t - 120°)$, $e_{c_1c_2} = E_m \sin (\omega t - 240°)$.

voltage. If the armature is stationary and the field is rotated, six leads are needed for the three independent phases. Since most apparatus using three-phase power does not require complete electrical isolation of the phases, two methods of interconnecting the three phases are used in which the number of leads from the source is reduced. The voltages

may be subtracted or added vectorially. Subtraction of the generated voltages gives the wye (or star) connection and addition gives the delta connection. The names of the two connections are derived, in part, from the resemblance of the schematic circuit diagrams to the letters "Y" and "Δ."

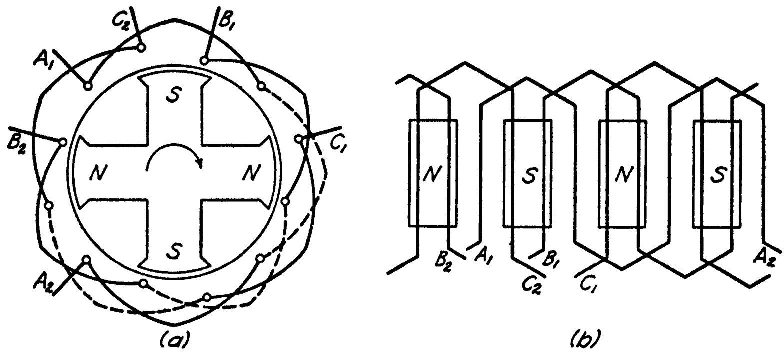


FIG. 9–12. (a) Coil displacement for a three-phase generator. (b) Developed winding showing the coil sides.

4. Wye or Star Connection. If the three windings, Fig. 10–12a, producing the three generated voltages $E_{A_1A_2}$, $E_{B_1B_2}$, and $E_{C_1C_2}$ are connected electrically at A_1, B_1, and C_1, the source becomes a wye- or star-connected three-phase supply with four leads, A_2, B_2, C_2, and N (neutral) available. The voltages E_{NA_2}, E_{NB_2}, and E_{NC_2} are equal, respec-

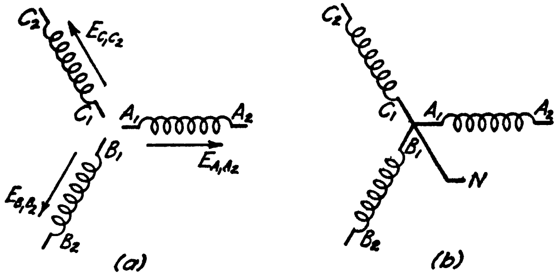


FIG. 10–12. (a) Three-phase generator coils. (b) Coils connected to give a three-phase four-wire system.

tively, to the phase voltages $E_{A_1A_2}$, $E_{B_1B_2}$, and $E_{C_1C_2}$. The generated voltages between lines ($E_{A_2B_2}$, $E_{B_2C_2}$, and $E_{C_2A_2}$) are obtained as shown in Fig. 11–12 by subtracting phase voltages.

$$\bar{E}_{C_2A_2} = -\bar{E}_{NC_2} + \bar{E}_{NA_2} = \bar{E}_{C_2N} + \bar{E}_{NA_2}$$
$$\bar{E}_{A_2B_2} = -\bar{E}_{NA_2} + \bar{E}_{NB_2} = \bar{E}_{A_2N} + \bar{E}_{NB_2}$$
$$\bar{E}_{B_2C_2} = -\bar{E}_{NB_2} + \bar{E}_{NC_2} = \bar{E}_{B_2N} + \bar{E}_{NC_2}$$

The voltmeter measures terminal not generated voltage. The terminal voltages $\bar{V}_{C_2A_2}$, $\bar{V}_{A_2B_2}$, and $\bar{V}_{B_2C_2}$ will be indicated on the vector diagram at an angular displacement of 180 electrical degrees from the generated voltages $\bar{E}_{C_2A_2}$, $\bar{E}_{A_2B_2}$, and $\bar{E}_{B_2C_2}$, respectively.

The sequence of the phase voltages (line to neutral voltages) is $A_2B_2C_2$, and the phase sequence of the line voltages is also $A_2B_2C_2$. The phase sequence can be expressed by the order of the first letter or the last letter in the subscripts of the currents or voltages. For example, if the line voltages are $E_{C_2A_2}$, $E_{A_2B_2}$, and $E_{B_2C_2}$, the phase sequence may be expressed as either $C_2A_2B_2$ (first letters) or $A_2B_2C_2$ (last letters) and is, in either expression, an ABC sequence. The line voltages are $\sqrt{3}$ times greater than the phase voltages. To prove this, assume the voltages E_{NA_2}, E_{NB_2}, and E_{NC_2} to have unit magnitude and have the following complex expressions.

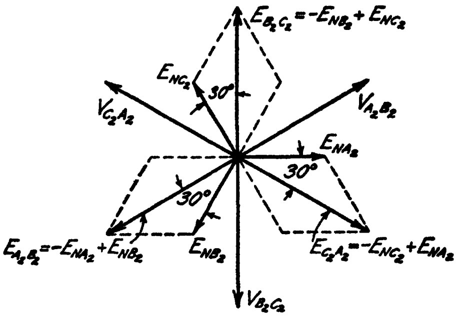


FIG. 11–12. The line voltages of a wye system obtained by subtraction of the phase voltages. Sequence ABC.

$$\bar{E}_{NA_2} = 1 + j0$$

$$\bar{E}_{NB_2} = -0.5 - j0.866$$

$$\bar{E}_{NC_2} = -0.5 + j0.866$$

$$\bar{E}_{C_2A_2} = -(-0.5 + j0.866) + (1 + j0) = 1.5 - j0.866$$

$$E_{C_2A_2} = \sqrt{(1.5)^2 + (0.866)^2} = \sqrt{3}$$

$$\bar{E}_{A_2B_2} = -(1 + j0) + (-0.5 - j0.866) = -1.5 - j0.866$$

$$E_{A_2B_2} = \sqrt{(1.5)^2 + (0.866)^2} = \sqrt{3}$$

$$\bar{E}_{B_2C_2} = -(-0.5 - j0.866) + (-0.5 + j0.866) = 0 + j1.732$$

$$E_{B_2C_2} = \sqrt{(0)^2 + (1.732)^2} = \sqrt{3}$$

In Fig. 11–12, the three-phase voltages are connected in star, and the three-line voltages (generated and terminal) are equal and are displaced 120° from each other. A 30° displacement exists between a line voltage and a phase voltage, and the line voltage is equal to $\sqrt{3}$ times the phase voltage.

If this three-phase source is connected to a three-phase symmetrical load, equal currents will flow in the three lines. In Fig. 12–12, it will be observed that there is a current in each line and a phase current in

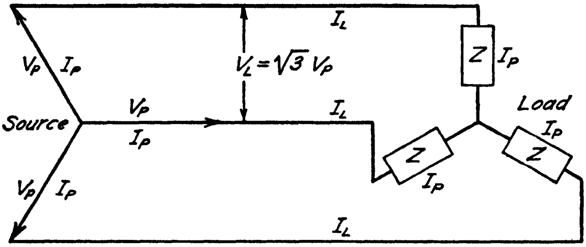


FIG. 12–12. Relationship of line and phase voltages in a three-phase wye connection.

the generator and in the load. The line current and current to neutral are the same. The relationship between phase (or line to neutral) and line values of currents and voltages for a balanced wye (or star) system can be summarized as

Phase (or Line to Neutral) Values	Line Values
$V_p = \dfrac{V_L}{\sqrt{3}}$	$V_L = \sqrt{3}\, V_p$
$I_p = I_L$	$I_L = I_p$

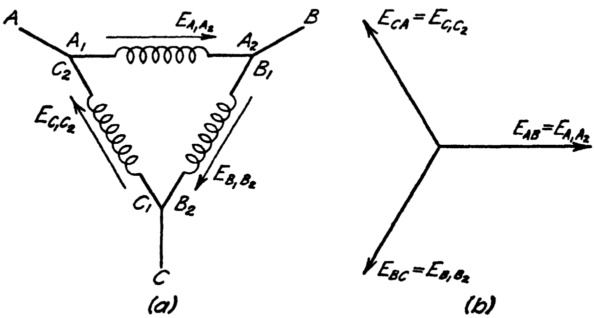


FIG. 13–12. (*a*) Connection of three windings in delta. (*b*) The vector diagram of the line values of generated voltages. Phase sequence *ABC*.

5. Delta Connections. If the three windings of a generator are connected as shown in Fig. 13–12*a*, the windings form a delta. The three main leads to this delta winding are *A*, *B*, and *C*. The voltages between the main leads are each equal in magnitude and direction to correspond-

ing coil voltages. The voltage $\bar{E}_{AB} = \bar{E}_{A_1A_2}$, $\bar{E}_{BC} = \bar{E}_{B_1B_2}$, and $\bar{E}_{CA} = \bar{E}_{C_1C_2}$. If a three-phase load is connected to this delta-connected source (Fig. 14–12), the current in each line is equal to the currents

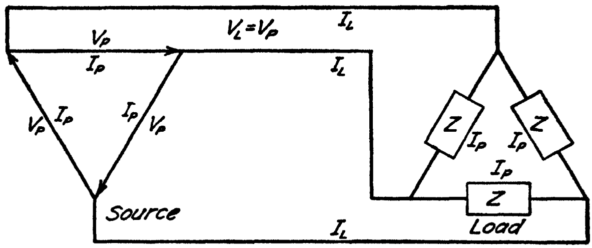


FIG. 14–12. Three-phase delta load connected to a three-phase delta-connected source.

from two generator phases added vectorially. In Fig. 15–12, a vector diagram shows the relationship between the phase currents and line currents. The line currents in every instance equal the vector difference of two phase currents and, if the system is balanced. they are $\sqrt{3}$ times the phase currents.

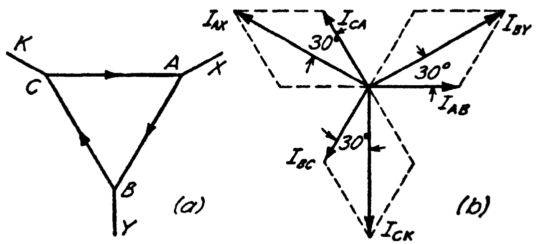


FIG. 15–12. (a) Circuit connections showing phase currents and line currents of a three-phase, delta-connected alternator. (b) Vector diagram showing the relationship of the phase and line currents.

The line currents can be obtained by using Kirchhoff's Law for currents at the junctions A, B, and C.

$$\bar{I}_{XA} + \bar{I}_{CA} + \bar{I}_{BA} = 0 \quad \text{and} \quad \bar{I}_{AX} = \bar{I}_{CA} + \bar{I}_{BA}$$
$$\bar{I}_{YB} + \bar{I}_{AB} + \bar{I}_{CB} = 0 \quad \text{and} \quad \bar{I}_{BY} = \bar{I}_{AB} + \bar{I}_{CB}$$
$$\bar{I}_{KC} + \bar{I}_{AC} + \bar{I}_{BC} = 0 \quad \text{and} \quad \bar{I}_{CK} = \bar{I}_{AC} + \bar{I}_{BC}$$

As previously shown in the relationship of the voltages for the star connection, if the phase current vectors are assumed to be one unit in length and have the following complex expressions:

$$\bar{I}_{AB} = 1 + j0$$
$$\bar{I}_{BC} = -0.5 - j0.866$$
$$\bar{I}_{CA} = -0.5 + j0.866$$

the values of the line currents become:

$$\bar{I}_{AX} = (-0.5 + j0.866) + (-1 + j0) = -1.5 + j0.866$$

$$I_{AX} = \sqrt{(1.5)^2 + (0.866)^2} = \sqrt{3}$$

$$\bar{I}_{BY} = (1 + j0) + (0.5 + j0.866) = 1.5 + j0.866$$

$$I_{BY} = \sqrt{(1.5)^2 + (0.866)^2} = \sqrt{3}$$

$$\bar{I}_{CK} = (0.5 - j0.866) + (-0.5 - j0.866) = 0 - j1.732$$

$$I_{CK} = \sqrt{(0)^2 + (1.732)^2} = \sqrt{3}$$

In Fig. 15–12*b*, there is a 30° displacement between the line and phase currents. The relationship of the delta and line value of currents and voltages for a balanced delta system can be summarized as:

Phase (or Delta) Values	Line Values
$V_p = V_L$	$V_L = V_p$
$I_p = \dfrac{I_L}{\sqrt{3}}$	$I_L = \sqrt{3}\, I_p$

6. Summary of Wye (or Star) and Delta Connections. The previous discussions have shown that a three-phase system may be connected in either star or delta. It is difficult to show that one type of connection is superior to the other in power transmission. However, the star connection is used in nearly every commercial generator today, because it provides a point of grounding for relay protection and because it allows the minimum voltage per coil for a desired line voltage. The distribution of power at the present time is about equally divided between the three-phase, three-wire and three-phase, four-wire systems. Most operating companies adopt one or the other and design their system protections accordingly.

7. Three-Phase Loads: General. A three-phase system is loaded by connecting the loads in either star or delta, and the loads may be either balanced or unbalanced. A balanced load is one in which the line voltages are equal and the loading per phase is identical, making the currents per phase equal in magnitude and displaced 120°. An unbalanced load may be produced by a symmetrical load on unbalanced line voltages or by balanced line voltages with a different load per phase.

A three-phase load connected in star may be connected to a supply by three wires or four wires, depending upon the use of the neutral of the source. If the neutral is used, the system then becomes three single-phase loads, each connected to a separate phase of the source. Both types of connections will be illustrated by examples.

8. Balanced Star System. As an example (Fig. 16–12) of a balanced, star-connected load connected to a star-connected source assume the following values for the impedances and voltages.

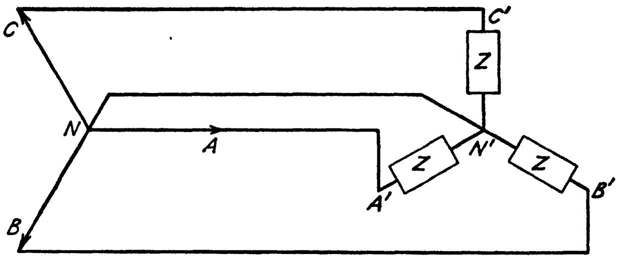


FIG. 16–12. A three-phase balanced load connected to a three-phase, star-connected source.

Phase sequence ABC:

$$\bar{V}_{AN} = 100 + j0 \qquad \bar{Z}_{A'N'} = 3 + j4$$

$$\bar{V}_{BN} = -50 - j86.6 \qquad \bar{Z}_{B'N'} = 3 + j4$$

$$\bar{V}_{CN} = -50 + j86.6 \qquad \bar{Z}_{C'N'} = 3 + j4$$

Find the current in each line, load impedance, and phase of the source.

Since this system is balanced, the phase voltages are equal and displaced 120° and the load impedances are equal. Each phase of the source has an individual load upon it and, since the neutral is the return for each phase, each load can be solved as an independent circuit. Kirchhoff's Laws hold for each closed loop and junction point, as they do in the single-phase system. The equations for the three loops can be written as follows.

Loop $NCC'N'$: $$\bar{V}_{NC} + \bar{I}_{C'N'}\bar{Z}_{C'N'} = 0$$

Substituting values,

$$(50 - j86.6) + \bar{I}_{C'N'}(3 + j4) = 0$$

$$\bar{I}_{C'N'} = \frac{-50 + j86.6}{3 + j4} = 7.85 + j18.39$$

Loop $NBB'N'$: $\bar{V}_{NB} + \bar{I}_{B'N'}\bar{Z}_{B'N'} = 0$

Substituting values,

$$(50 + j86.6) + \bar{I}_{B'N'}(3 + j4) = 0$$

$$\bar{I}_{B'N'} = \frac{-50 - j86.6}{3 + j4} = -19.85 - j2.39$$

Loop $NAA'N'$: $\bar{V}_{NA} + \bar{I}_{A'N'}\bar{Z}_{A'N'} = 0$

Substituting values,

$$(-100 + j0) + \bar{I}_{A'N'}(3 + j4) = 0$$

$$\bar{I}_{A'N'} = \frac{100 + j0}{3 + j4} = 12 - j16$$

The three currents, $I_{A'N'}$, $I_{B'N'}$, and $I_{C'N'}$ have been determined and these same currents exist in the respective lines and phases of the source. If the currents flowing toward junction N' are added together, the current $I_{NN'}$ is zero.

$$\bar{I}_{A'N'} + \bar{I}_{B'N'} + \bar{I}_{C'N'} + \bar{I}_{NN'} = 0$$

Substituting values,

$$(12 - j16) + (-19.85 - j2.39) + (7.85 + j18.39) + \bar{I}_{NN'} = 0$$

$$\bar{I}_{NN'} = 0$$

The lead NN' carries no current when the system is balanced, and the difference of potential between N and N' is zero. Therefore, if the system is balanced, the use of the neutral wire is optional.

9. Unbalanced Star System. If the circuit of Fig. 16–12 is considered with unbalanced impedances as loads, the solution is identical with that of the balanced system except that the neutral will carry a current. Assume the following values for voltages and impedances.

Phase sequence ABC:

$$\bar{V}_{AN} = 100 + j0 \qquad \bar{Z}_{A'N'} = 5 + j0$$

$$\bar{V}_{BN} = -50 - j86.6 \qquad \bar{Z}_{B'N'} = 3 + j4$$

$$\bar{V}_{CN} = -50 + j86.6 \qquad \bar{Z}_{C'N'} = 3 + j4$$

If the same procedure followed for the balanced system is used, the currents will be

$$\bar{I}_{A'N'} = 20 + j0$$

$$\bar{I}_{B'N'} = -19.85 - j2.39$$

$$\bar{I}_{C'N'} = 7.85 + j18.39$$

The neutral line current $\bar{I}_{NN'}$ will be found from

$$\bar{I}_{A'N'} + \bar{I}_{B'N'} + \bar{I}_{C'N'} + \bar{I}_{NN'} = 0$$

Substituting values,

$$(20 + j0) + (-19.85 - j2.39) + (7.85 + j18.39) + \bar{I}_{NN'} = 0$$

$$\bar{I}_{NN'} = -8 - j16$$

If the loads are not balanced, the neutral current is the vector sum of the currents in the other lines. The neutral current, therefore, becomes the current produced by the unbalanced condition. If the neutral wire is removed, the system is not the same as it is with the neutral connected, and the voltage across each impedance is not the voltage per phase of the source. The solution of a circuit of this type will be shown as an example in the general application of Kirchhoff's Laws to polyphase circuits.

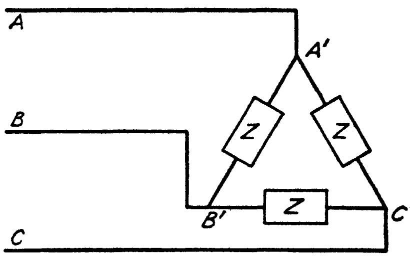


FIG. 17–12. Balanced delta load connected to a three-phase source.

10. Balanced Delta System. In Fig. 17–12, three impedances are connected in delta across a balanced three-phase source of supply. As in the star system, the voltage across each impedance is a definite source voltage of one phase. The solution, therefore, of the current per phase is obtained in the same manner as it is in the star system. Assume the following values for the voltages and impedances of Fig. 17–12

$$\bar{V}_{AB} = 100 + j0 \qquad \bar{Z}_{A'B'} = 3 + j4$$

$$\bar{V}_{BC} = -50 - j86.6 \qquad \bar{Z}_{B'C'} = 3 + j4$$

$$\bar{V}_{CA} = -50 + j86.6 \qquad \bar{Z}_{C'A'} = 3 + j4$$

The current for each impedance can be obtained, as in the star system, by the use of Kirchhoff's Laws for each loop.

Loop $ABB'A'$: $\qquad \bar{V}_{AB} + \bar{I}_{B'A'}\bar{Z}_{B'A'} = 0$

Substituting values,

$$(100 + j0) + \bar{I}_{B'A'}(3 + j4) = 0$$

$$\bar{I}_{A'B'} = \frac{100 + j0}{3 + j4} = 12 - j16$$

Loop $BCC'B'$: $$\bar{V}_{BC} + \bar{I}_{C'B'}\bar{Z}_{C'B'} = 0$$

Substituting values,

$$(-50 - j86.6) + \bar{I}_{C'B'}(3 + j4) = 0$$

$$\bar{I}_{B'C'} = \frac{-50 - j86.6}{3 + j4} = -19.85 - j2.39$$

Loop $CAA'C'$: $$\bar{V}_{CA} + \bar{I}_{A'C'}\bar{Z}_{A'C'} = 0$$

Substituting values,

$$(-50 + j86.6) + \bar{I}_{A'C'}(3 + j4) = 0$$

$$\bar{I}_{C'A'} = \frac{-50 + j86.6}{3 + j4} = 7.85 + j18.39$$

These values are the currents for each impedance in the load, and the line currents $I_{AA'}$, $I_{BB'}$, and $I_{CC'}$ are obtained by using Kirchhoff's Second Law for currents at the junctions A', B', and C'.

Junction A': $$\bar{I}_{AA'} + \bar{I}_{B'A'} + \bar{I}_{C'A'} = 0$$

$$\bar{I}_{AA'} = \bar{I}_{A'B'} + \bar{I}_{A'C'}$$

Substituting values,

$$\bar{I}_{AA'} = (12 - j16) + (-7.85 - j18.39) = 4.15 - j34.39$$

Junction B': $$\bar{I}_{BB'} + \bar{I}_{A'B'} + \bar{I}_{C'B'} = 0$$

$$\bar{I}_{BB'} = \bar{I}_{B'A'} + \bar{I}_{B'C'}$$

Substituting values,

$$\bar{I}_{BB'} = (-12 + j16) + (-19.85 - j2.39) = -31.85 + j13.61$$

Junction C': $$\bar{I}_{CC'} + \bar{I}_{B'C'} + \bar{I}_{A'C'} = 0$$

$$\bar{I}_{CC'} = \bar{I}_{C'B'} + \bar{I}_{C'A'}$$

Substituting values,

$$\bar{I}_{CC'} = (19.85 + j2.39) + (7.85 + j18.39) = 27.70 + j20.78$$

The vector sum of the line currents is zero.

$$\bar{I}_{AA'} + \bar{I}_{BB'} + \bar{I}_{CC'} = 0$$

Substituting values,

$$(4.15 - j34.39) + (-31.85 + j13.61) + (27.70 + j20.78) = 0$$

and

$$0 = 0$$

The line currents equal $\sqrt{3}$ times the phase currents for a balanced system. The current for each impedance is 20 amperes, and the line currents are 34.64 amperes in this case.

11. Unbalanced Delta System. If unequal impedances are connected in delta (Fig. 17–12), the currents per impedance are not equal but are obtained in the same way that they would be if the impedances were equal. Assume the following values for the voltages and impedances:

$$\bar{V}_{AB} = 100 + j0 \qquad \bar{Z}_{A'B'} = 5 + j0$$

$$\bar{V}_{BC} = -50 - j86.6 \qquad \bar{Z}_{B'C'} = 3 + j4$$

$$\bar{V}_{CA} = -50 + j86.6 \qquad \bar{Z}_{C'A'} = 3 + j4$$

From these values,

$$\bar{I}_{A'B'} = 20 + j0$$

$$\bar{I}_{B'C'} = -19.85 - j2.39$$

and

$$\bar{I}_{C'A'} = 7.85 + j18.39$$

The line currents will not be equal in magnitude for this condition because the individual phase currents, though equal in magnitude, are not displaced 120° from each other.

$$\bar{I}_{AA'} = (20 + j0) + (-7.85 - j18.39) = 12.15 - j18.39$$

$$\bar{I}_{BB'} = (-20 + j0) + (-19.85 - j2.39) = -39.85 - j2.39$$

$$\bar{I}_{CC'} = (19.85 + j2.39) + (7.85 + j18.39) = 27.70 + j20.78$$

However, the vector sum of the three line currents is zero (all in the same direction), regardless of the fact that the three currents are not equal.

$$\bar{I}_{AA'} + \bar{I}_{BB'} + \bar{I}_{CC'} = 0$$

$$(12.15 - j18.39) + (-39.85 - j2.39) + (27.70 + j20.78) = 0$$

and

$$0 = 0$$

The solution of an unbalanced system is obtained in the same manner as that of a balanced system. The fact that the individual impedances are each connected to a separate voltage (or its equivalent) makes it possible to solve for the individual phase currents.

12. Unbalanced Systems: Star and Delta. Three-phase systems may be unbalanced as to voltages or impedances and, when the line impedances are considered also, the system becomes complicated. The solution, however, is obtained by the same method used for the simple condi-

tions; namely, by the use of Kirchhoff's Laws. The fact that the voltages of a system may be displaced in time phase, with involved impedances connected to them, does not alter the application of these fundamental laws. To demonstrate the application to more involved circuits, two examples will be given, one showing an unbalanced star circuit and the other an unbalanced delta circuit. It must be remembered that, in every instance, the circuit must be reduced to its simplest form before Kirchhoff's Laws are applied.

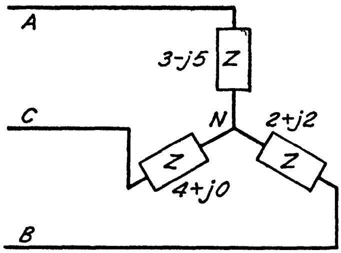


FIG. 18–12. Circuit diagram of an unbalanced wye-connected load connected to a three-phase voltage supply.

(a) *Star System.* The circuit indicated by Fig. 18–12 has the following voltage and impedance values, and the currents through each impedance are to be determined.

Phase sequence ABC:

$$\bar{V}_{AB} = 100 + j0 \qquad \bar{Z}_{AN} = 3 - j5$$

$$\bar{V}_{BC} = -50 - j86.6 \qquad \bar{Z}_{BN} = 2 + j2$$

$$\bar{V}_{CA} = -50 + j86.6 \qquad \bar{Z}_{CN} = 4 + j0$$

If the impedances are not balanced, the currents through the impedances will not be equal. However, the circuit can be solved by Kirchhoff's Laws in the same way that any single-phase circuit would be solved. Three equations are required since there are three unknown currents. The equations to be solved are:

Loop $ABNA$: $$\bar{V}_{AB} + \bar{I}_{BN}\bar{Z}_{BN} + \bar{I}_{NA}\bar{Z}_{NA} = 0$$

Loop $BCNB$: $$\bar{V}_{BC} + \bar{I}_{CN}\bar{Z}_{CN} + \bar{I}_{NB}\bar{Z}_{NB} = 0$$

Junction N: $$\bar{I}_{AN} + \bar{I}_{BN} + \bar{I}_{CN} = 0$$

If $\bar{I}_{CN}$ is eliminated from the voltage equations and the values for the voltages are substituted, the equations become

$$\bar{I}_{CN} = \bar{I}_{NB} + \bar{I}_{NA}$$

(a) $$(100 + j0) + \bar{I}_{BN}(2 + j2) + \bar{I}_{NA}(3 - j5) = 0$$

(b) $$(-50 - j86.6) + \bar{I}_{NB}(6 + j2) + \bar{I}_{NA}(4 + j0) = 0$$

Eliminating $\bar{I}_{NA}$,

$$(a')\quad (100 + j0)(4 + j0) + \bar{I}_{BN}(2 + j2)(4 + j0) + \bar{I}_{NA}(3 - j5)(4 + j0) = 0$$

$$(b')\quad (-50 - j86.6)(3 - j5) - \bar{I}_{BN}(6 + j2)(3 - j5) + \bar{I}_{NA}(3 - j5)(4 + j0) = 0$$

and simplifying,

$$(A)\quad 400 + j0 + \bar{I}_{BN}(8 + j8) + \bar{I}_{NA}(3 - j5)(4 + j0) = 0$$

$$(B)\quad (-583 - j9.8) - \bar{I}_{BN}(28 - j24) + \bar{I}_{NA}(3 - j5)(4 + j0) = 0$$

Subtracting,

$$-983 - j9.8 - \bar{I}_{BN}(36 - j16) = 0$$

$$\bar{I}_{BN} = \frac{-983 - j9.8}{36 - j16} = -22.7 - j10.36$$

Substituting the value of $\bar{I}_{BN}$ in equation (*b*),

$$(-50 - j86.6) + (22.7 + j10.36)(6 + j2) + \bar{I}_{NA}(4 + j0) = 0$$

$$(-50 - j86.6) + (115 + j109) + \bar{I}_{NA}(4 + j0) = 0$$

$$\bar{I}_{NA} = \frac{-65 - j22.4}{4 + j0} = -16.25 - j5.6$$

and from the relationship at junction N,

$$\bar{I}_{CN} + \bar{I}_{BN} + \bar{I}_{AN} = 0$$

$$\bar{I}_{CN} = (22.7 + j10.36) + (-16.25 - j5.6)$$

$$\bar{I}_{CN} = 6.45 + j4.76$$

The three currents (which are also phase currents) are

$$\bar{I}_{AN} = 16.25 + j5.6 \qquad I_{AN} = 17.18 \text{ amperes}$$

$$\bar{I}_{BN} = -22.7 - j10.36 \qquad I_{BN} = 24.98 \text{ amperes}$$

$$\bar{I}_{CN} = 6.45 + j4.76 \qquad I_{CN} = 8.03 \text{ amperes}$$

(*b*) *Delta System.* The circuit in Fig. 19–12 shows an unbalanced delta load, connected to a three-phase supply through lines having a given impedance. The currents in each load impedance as well as the line currents are found from the solution of six simultaneous equations.

Phase sequence ABC:

$$\bar{V}_{AB} = 100 + j0$$
$$\bar{V}_{BC} = -50 - j86.6$$
$$\bar{V}_{CA} = -50 + j86.6$$

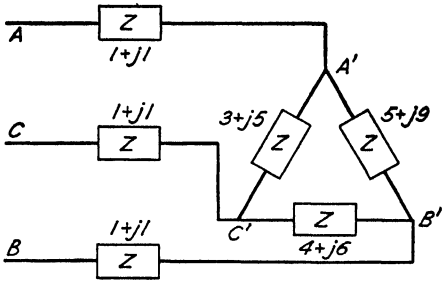


FIG. 19–12. Circuit diagram of a three-phase unbalanced delta load connected to a three-phase supply through lines having a given impedance.

The six equations necessary to obtain a solution can be written:

Loop $ABB'A'A$:

(a) $\bar{V}_{AB} + \bar{I}_{BB'}(1 + j1) + \bar{I}_{B'A'}(5 + j9) + \bar{I}_{A'A}(1 + j1) = 0$

Loop $BCC'B'B$:

(b) $\bar{V}_{BC} + \bar{I}_{CC'}(1 + j1) + \bar{I}_{C'B'}(4 + j6) + \bar{I}_{B'B}(1 + j1) = 0$

Loop $CAA'C'C$:

(c) $\bar{V}_{CA} + \bar{I}_{AA'}(1 + j1) + \bar{I}_{A'C'}(3 + j5) + \bar{I}_{C'C}(1 + j1) = 0$

Junction A':

(d) $\bar{I}_{AA'} + \bar{I}_{C'A'} + \bar{I}_{B'A'} = 0 \qquad \bar{I}_{AA'} = -\bar{I}_{C'A'} - \bar{I}_{B'A'}$

Junction B':

(e) $\bar{I}_{BB'} + \bar{I}_{A'B'} + \bar{I}_{C'B'} = 0 \qquad \bar{I}_{BB'} = -\bar{I}_{C'B'} - \bar{I}_{A'B'}$

Junction C':

(f) $\bar{I}_{CC'} + \bar{I}_{B'C'} + \bar{I}_{A'C'} = 0 \qquad \bar{I}_{CC'} = -\bar{I}_{B'C'} - \bar{I}_{A'C'}$

Substituting the values for $\bar{I}_{AA'}$, $\bar{I}_{BB'}$, and $\bar{I}_{CC'}$ in equations (a), (b), and (c), the following three equations, containing three unknown currents, are obtained.

(1) $\bar{V}_{AB} - \bar{I}_{C'B'}(1 + j1) - \bar{I}_{A'B'}(7 + j11) - \bar{I}_{A'C'}(1 + j1) = 0$

(2) $\bar{V}_{BC} + \bar{I}_{C'B'}(6 + j8) + \bar{I}_{A'B'}(1 + j1) - \bar{I}_{A'C'}(1 + j1) = 0$

(3) $\bar{V}_{CA} - \bar{I}_{C'B'}(1 + j1) + \bar{I}_{A'B'}(1 + j1) + \bar{I}_{A'C'}(5 + j7) = 0$

From (1) minus (2),

(4) $$(\bar{V}_{AB} - \bar{V}_{BC}) - \bar{I}_{C'B'}(7 + j9) - \bar{I}_{A'B'}(8 + j12) = 0$$

Multiplying (2) by $(5 + j7)$ and adding to [(3) multiplied by $(1 + j1)$]:

(5) $$[\bar{V}_{BC}(5 + j7) + \bar{V}_{CA}(1 + j1)] + \bar{I}_{C'B'}(-26 + j80) + \bar{I}_{A'B'}(-2 + j14) = 0$$

Now, substituting the values for the voltages in equations (4) and (5) and simplifying, the equations become

(6) $$(150 + j86.6) - \bar{I}_{C'B'}(7 + j9) - \bar{I}_{A'B'}(8 + j12) = 0$$

(7) $$(219.6 - j746.4) + \bar{I}_{C'B'}(-26 + j80) + \bar{I}_{A'B'}(-2 + j14) = 0$$

Eliminating $\bar{I}_{A'B'}$ by multiplying (6) by $(-2 + j14)$ and (7) by $(8 + j12)$ and adding, one equation is obtained. This equation contains only one unknown current:

(6a) $$(-1512.4 + j1926.8) - \bar{I}_{C'B'}(-140 + j80) - \bar{I}_{A'B'}(8 + j12)(-2 + j14) = 0$$

(7a) $$(10713.6 - j3336) + \bar{I}_{C'B'}(-1168 + j328) + \bar{I}_{A'B'}(8 + j12)(-2 + j14) = 0$$

Adding (6a) and (7a),

(8) $$(9201.2 - j1409.2) + \bar{I}_{C'B'}(-1028 + j248) = 0$$

and

$$\bar{I}_{C'B'} = \frac{-9201.2 + j1409.2}{-1028 + j248} = 8.7708 + j0.7451$$

Substituting in (6) the value of $\bar{I}_{C'B'}$,

(9) $$(150 + j86.6) - (8.7708 + j0.7451)(7 + j9) - \bar{I}_{A'B'}(8 + j12) = 0$$

(10) $$(150 + j86.6) - (54.6897 + j84.1529) - \bar{I}_{A'B'}(8 + j12) = 0$$

(11) $$(95.3103 + j2.4471) - \bar{I}_{A'B'}(8 + j12) = 0$$

and

$$\bar{I}_{A'B'} = \frac{-95.3103 - j2.4471}{8 + j12} = 3.8069 - j5.4045$$

Substituting the values of $\bar{I}_{C'B'}$ and $\bar{I}_{A'B'}$ in (3),

(12) $$(-50 + j86.6) - (8.7708 + j0.7451)(1 + j1) + (3.8069 - j5.4045)(1 + j1) + \bar{I}_{A'C'}(5 + j7) = 0$$

(13) $$(-50 + j86.6) - (8.027 + j9.5159) + (9.2114 - j1.5976) + \bar{I}_{A'C'}(5 + j7) = 0$$

(14) $$(-48.8143 + j75.4865) + \bar{I}_{A'C'}(5 + j7) = 0$$

and

$$\bar{I}_{A'C'} = \frac{48.8143 - j75.4865}{5 + j7} = -3.8423 - j9.7180$$

Solving for the line currents,

(15) $$\bar{I}_{AA'} = (-3.8423 - j9.7180) + (3.8069 - j5.4045)$$
$$\bar{I}_{AA'} = -0.0354 - j15.1225$$

(16) $$\bar{I}_{BB'} = (-8.7708 - j0.7451) + (-3.8069 + j5.4045)$$
$$\bar{I}_{BB'} = -12.5777 + j4.6594$$

(17) $$\bar{I}_{CC'} = (8.7708 + j0.7451) + (3.8423 + j9.7180)$$
$$\bar{I}_{CC'} = 12.6131 + j10.4631$$

The six currents in this unbalanced circuit are:

Phase currents:

$\bar{I}_{A'B'} = 3.8069 - j5.4045$ $\qquad I_{A'B'} = 6.6$ amperes

$\bar{I}_{B'C'} = -8.7708 - j0.7451$ $\qquad I_{B'C'} = 8.79$ amperes

$\bar{I}_{C'A'} = 3.8423 + j9.7180$ $\qquad I_{C'A'} = 10.32$ amperes

Line currents:

$\bar{I}_{AA'} = -0.0354 - j15.1225$ $\qquad I_{AA'} = 15.13$ amperes

$\bar{I}_{BB'} = -12.5777 + j4.6594$ $\qquad I_{BB'} = 13.38$ amperes

$\bar{I}_{CC'} = 12.6131 + j10.4631$ $\qquad I_{CC'} = 16.35$ amperes

The solutions of these two types of unbalanced three-phase circuits have been obtained in the same manner that any network solution would be obtained. The fundamental Kirchhoff's Laws are not restricted to any particular type of circuit and their use in polyphase circuits is a great aid in obtaining solutions. It must be remembered, however, that the circuits should be reduced to the simplest practical form before using Kirchhoff's Laws. This reduction decreases the amount of labor required to obtain a complete solution.

13. Equivalent Wye and Delta Three-Phase Systems. When solving three-phase circuit problems, it is often desirable to replace one circuit by an equivalent circuit. A wye-connected load may be replaced by its equivalent delta-connected load, and a delta-connected load by its

equivalent wye-connected load. An equivalent load is one which will require the same line currents with the same line voltages impressed and also maintain the same phase relationship between the line currents and line voltages as existed in the original load.

For balanced conditions (all impedances equal and having the same power factor) the substitution of an equivalent circuit is comparatively simple. As an example, assume that a balanced wye load is to be replaced by its equivalent delta load. For the wye load, the impedance per phase is

$$Z_p = \frac{V_p}{I_p} \quad \text{or} \quad Z_p = \frac{V_L}{\sqrt{3}\, I_L}$$

For the delta load, the impedance per phase is

$$Z_p = \frac{V_p}{I_p} \quad \text{or} \quad Z_p = \frac{\sqrt{3}\, V_L}{I_L}$$

The ratio of the delta phase impedance to the wye phase impedance becomes

$$\frac{Z_\Delta}{Z_y} = \frac{\dfrac{\sqrt{3}\, V_L}{I_L}}{\dfrac{V_L}{\sqrt{3}\, I_L}} = 3$$

Also the ratio of the wye phase impedance to the delta phase impedance becomes

$$\frac{Z_y}{Z_\Delta} = \frac{1}{3}$$

For unbalanced conditions (impedances not equal and not having the same power factor) the substitution of one system for the other is not

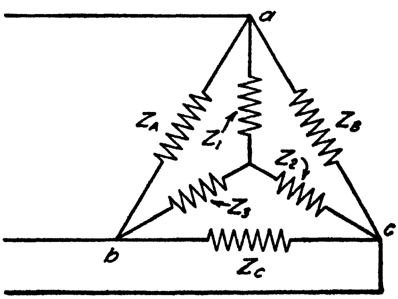


FIG. 20–12. Equivalent wye-to-delta or delta-to-wye transformation circuit.

too easily made. If the wye-connected impedances and delta-connected impedances of Fig. 20–12 are to be equivalent, measurements made

between the line terminals for both circuits must be identical. Therefore,

$$\bar{Z}_{ab} = \bar{Z}_1 + \bar{Z}_3 = \frac{\bar{Z}_A(\bar{Z}_B + \bar{Z}_C)}{\bar{Z}_A + \bar{Z}_B + \bar{Z}_C}$$

$$\bar{Z}_{bc} = \bar{Z}_2 + \bar{Z}_3 = \frac{\bar{Z}_C(\bar{Z}_A + \bar{Z}_B)}{\bar{Z}_A + \bar{Z}_B + \bar{Z}_C}$$

$$\bar{Z}_{ca} = \bar{Z}_1 + \bar{Z}_2 = \frac{\bar{Z}_B(\bar{Z}_A + \bar{Z}_C)}{\bar{Z}_A + \bar{Z}_B + \bar{Z}_C}$$

When these equations are solved for the wye-connected impedances in terms of the delta-connected impedances the results are

$$\bar{Z}_1 = \frac{\bar{Z}_A\bar{Z}_B}{\bar{Z}_A + \bar{Z}_B + \bar{Z}_C}$$

$$\bar{Z}_2 = \frac{\bar{Z}_B\bar{Z}_C}{\bar{Z}_A + \bar{Z}_B + \bar{Z}_C}$$

$$\bar{Z}_3 = \frac{\bar{Z}_A\bar{Z}_C}{\bar{Z}_A + \bar{Z}_B + \bar{Z}_C}$$

If the same equations are solved for the delta-connected impedances in terms of the wye-connected impedances the values become

$$\bar{Z}_A = \frac{\bar{Z}_1\bar{Z}_2 + \bar{Z}_2\bar{Z}_3 + \bar{Z}_3\bar{Z}_1}{\bar{Z}_2}$$

$$\bar{Z}_B = \frac{\bar{Z}_1\bar{Z}_2 + \bar{Z}_2\bar{Z}_3 + \bar{Z}_3\bar{Z}_1}{\bar{Z}_3}$$

$$\bar{Z}_C = \frac{\bar{Z}_1\bar{Z}_2 + \bar{Z}_2\bar{Z}_3 + \bar{Z}_3\bar{Z}_1}{\bar{Z}_1}$$

It must be remembered that for any conversion from delta to equivalent wye or from wye to equivalent delta the power consumed by the substituted circuit must equal the power consumed by the original circuit.

PROBLEMS

1–12. Three coils connected in series are spaced 15 electrical degrees apart. The coil sides are displaced 150 electrical degrees and each coil side generates a sine wave of voltage of 10 volts effective value. What is the total voltage generated?

2–12. An alternator supplies two separate voltages as follows: $\bar{V}_{A_1A_2} = 80 + j60$ and $\bar{V}_{B_1B_2} = 60 - j80$. These voltages may be connected in series. Determine the current in a 10-ohm impedance if a voltage of (*a*) $\bar{V}_{B_2A_2}$ or (*b*) $\bar{V}_{B_1A_2}$ is impressed.

3–12. An alternator is generating a 3-phase voltage. The voltages to neutral are $\bar{V}_{AN} = 100 + j0$; $\bar{V}_{BN} = -50 - j86.6$; $\bar{V}_{CN} = -50 + j86.6$. What are the complex expressions for the voltages $\bar{V}_{AB}$, $\bar{V}_{BC}$, and $\bar{V}_{CA}$? Draw a vector diagram showing all voltages.

4–12. The three line voltages of a 3-phase, 4-wire system are $\bar{V}_{AB} = 0 + j100$; $\bar{V}_{BC} = 86.6 - j50$; $\bar{V}_{CA} = -86.6 - j50$. What are the voltages $\bar{V}_{NA}$, $\bar{V}_{NB}$, and $\bar{V}_{NC}$?

5–12. A balanced 3-phase load is connected in delta. The impedance of each phase is $8 - j6$ and the line voltage is 125 volts. Determine the phase currents and line currents.

6–12. A 250-volt, 3-phase balanced source of *ABC* sequence supplies power to a balanced delta load. $\bar{V}_{AB} = 250 + j0$ and $\bar{I}_{AB} = 8 - j6$. Determine the complex expression for each line current and phase current. Draw a complete vector diagram.

7–12. A balanced wye load of $5 + j5$ ohms per phase is connected to a 3-phase 3-wire balanced source. The line voltage is 100 volts. Determine the current per phase.

8–12. If the phase sequence of Prob. 7 is *ACB* with $\bar{V}_{AB} = 100 + j0$, determine the complex expressions for $\bar{I}_{AN}$, $\bar{I}_{BN}$, and $\bar{I}_{CN}$. Draw a complete vector diagram.

9–12. A 100-volt, 3-phase balanced source is connected to a delta load. The load impedances are $\bar{Z}_{AB} = 4 - j3$; $\bar{Z}_{BC} = 0 - j5$; $\bar{Z}_{CA} = 5 + j0$. If $\bar{V}_{AB} = 0 - j100$ and the phase sequence is *ACB* determine the complex expression for each line current.

10–12. A 3-phase, 3-wire, 230-volt balanced source supplies power to a 3-phase balanced load. The line currents are 30 amp each and the system power factor is 0.8 lag. Give the values of each phase impedance (complex form) if the load is considered to be (*a*) delta connected and (*b*) wye connected.

11–12. Three impedances, $\bar{Z}_{AB} = 8.66 + j5$; $\bar{Z}_{BC} = 10 + j0$; $\bar{Z}_{CA} = 0 - j10$, constitute a delta-connected load on a balanced 3-phase system of voltages of *CBA* phase sequence where $\bar{V}_{AB} = 86.6 + j50$. Determine the three phase currents and the three line currents.

12–12. A 3-phase, 4-wire source with balanced voltages supplies power to a wye load. The phase sequence is *ACB* and the voltage $\bar{V}_{AN} = 100 + j0$. The impedances are $\bar{Z}_{AN} = 3 + j4$, $\bar{Z}_{BN} = 5 + j0$, $\bar{Z}_{CN} = 2.5 - j4.33$. What would be the readings of ammeters connected in the four lines to this load?

13–12. Three coil voltages, $\bar{V}_{NB} = 100 + j0$, $\bar{V}_{NC} = 70 - j70$, $\bar{V}_{NA} = -50 + j50$, are to be connected in *ABC* sequence. Determine the line voltages for a wye connection.

14–12. Three impedances, $\bar{Z}_{AN} = 5 - j5$, $\bar{Z}_{BN} = 5 + j5$, $\bar{Z}_{CN} = 5 + j0$, connected in wye, are supplied power from a 3-phase balanced voltage source of sequence *ACB* where $\bar{V}_{AB} = 100 + j0$. Determine the phase voltages and currents. Draw the complete vector diagram.

15–12. A 3-phase, balanced, 100-volt supply of *ABC* sequence is connected to a wye load and a delta load operating in parallel. Each wye impedance is $5 + j0$. The delta impedances are $\bar{Z}_{AB} = 4 + j3$, $\bar{Z}_{BC} = 0 - j10$, $\bar{Z}_{CA} = 10 + j0$. What is the complex expression for the three line currents $\bar{I}_{A'A}$, $\bar{I}_{B'B}$, and $\bar{I}_{C'C}$ if $\bar{V}_{AB} = 100 + j0$? Draw a vector diagram showing all voltages and currents.

CHAPTER 13

POWER IN POLYPHASE CIRCUITS

The power in any polyphase system is the algebraic summation of the average power in the individual phases. It has already been shown that power is *a scalar* and *not a vector quantity.* If it were possible to place a wattmeter in each phase of a system, the sum of the various wattmeter readings would be the total power but, because it is seldom practical to isolate the phases of a system, other methods are used in measuring the power of a polyphase system.

In any system the total power can be determined with (n–1) meters (Blondel's Theorem), where n represents the number of wires. For the three-phase, three-wire and the two-phase, three-wire systems, two meters are sufficient but, if the system is a four-wire system, three meters are required. In a two-phase system (not interconnected), though there are four wires, the two phases are essentially two single-phase systems and two meters will measure the power. The two-phase interconnected system also has four wires but, here, three meters are needed to determine the power. Since two-phase systems are obsolete, even though some are still in operation, and systems with more than three phases are used only for special applications, the measurement of power in a polyphase system will be confined to the three-phase problem.

1. Power in the Balanced Three-Phase System. Since all electrical apparatus is built symmetrically, the balanced three-phase system is the one usually considered in theory; however, it is not practical to assume that the system voltages or currents on any distribution system are balanced. It is best, therefore, to assume an unbalanced system when making actual measurements of power. With some apparatus, a 1 to 2 per cent unbalance of voltage will cause a 10 per cent unbalance in the currents of the polyphase circuit.

In delta and wye connections of balanced three-phase systems, the currents and voltages are

Delta	Wye
$I_L = \sqrt{3}\, I_p$	$I_L = I_p$
$V_L = E_p$	$V_L = \sqrt{3}\, E_p$

The power in each load or generator coil of either of the above connections will be

$$P_p = V_p I_p \cos\theta_p \tag{a–13}$$

When the line voltage and current values are substituted so that the power per phase will be expressed in these values, equation (a–13) takes the form

Delta	Wye
$P_p = V_L \dfrac{I_L}{\sqrt{3}} \cos\theta_p$	$P_p = \dfrac{V_L}{\sqrt{3}} I_L \cos\theta_p$

The power per phase, regardless of the connection, is

$$P_p = \frac{1}{\sqrt{3}} V_L I_L \cos\theta_p$$

and, since the system is balanced, all three phases will be alike and the total power will be

$$P = 3 \times \frac{1}{\sqrt{3}} V_L I_L \cos\theta_p$$

The expression commonly used for three-phase power is

$$P = \sqrt{3}\, VI \cos\theta$$

where V is the line voltage, I is the line current, and the power factor is that of the load impedance in each phase.

The calculation of power in a balanced three-phase system is no more difficult than the calculation of currents and voltages of the system.

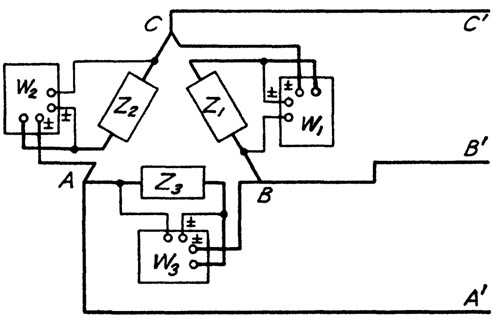


FIG. 1–13. Measurement of three-phase power in a delta system, using three wattmeters.

2. Measurement of Power in Balanced Three-Phase Systems. Figures 1–13 and 2–13 show the most direct method of measuring three-phase power. In each illustration, it has been assumed that it is pos-

sible to place meters in the delta or wye connection. In practice this is often impossible, for it is seldom that the delta winding of the load can be opened, or that the neutral of the wye is accessible.

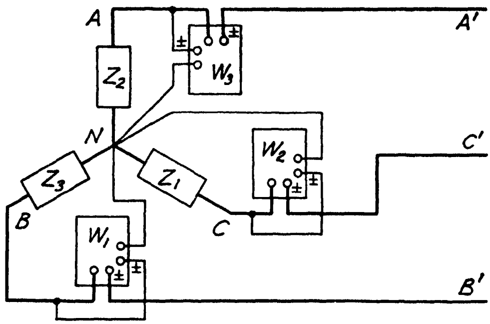


FIG. 2–13. Measurement of three-phase power in a wye system, using three wattmeters.

If the power indicated by each of the three meters is added, the sum will be the power consumed by the load. Since the measurements are actually taken in each phase, the three-meter method is applicable to the unbalanced system as well as to the balanced system. In a balanced system, the power can be measured with one meter, the reading of the single meter being multiplied by three to determine the actual power.

In practice, particularly when measuring small loads, meters having the same characteristics should be used because meters of different

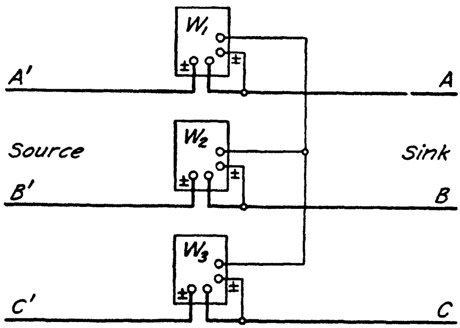


FIG. 3–13. Measuring the power in any three-phase system, using three wattmeters having a common potential neutral.

impedances cause unbalanced conditions, and make meter load corrections difficult.

Figures 3–13 and 4–13 show the connections of meters on a three-phase system in which the power readings may be determined from the

line currents and voltages. If the meters are similar, the readings will be the same for a balanced load and the sum will be the total power but, if the meters have different impedances, the meter readings will

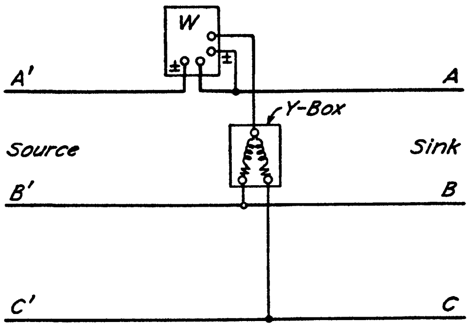


FIG. 4–13. Measuring the power in a balanced three-phase system, using one wattmeter and an impedance (wye) box.

differ, though their sum will still be the total power. In Fig. 4–13, two of the meters are replaced by a wye box having impedances equal to the potential circuit of the meter used, and the total power is three times the meter reading.

3. Measurement of Power in a Balanced Three-Phase System with Two Wattmeters. It has been stated that the power in a three-phase three-wire system may be measured by two wattmeters. Consider the

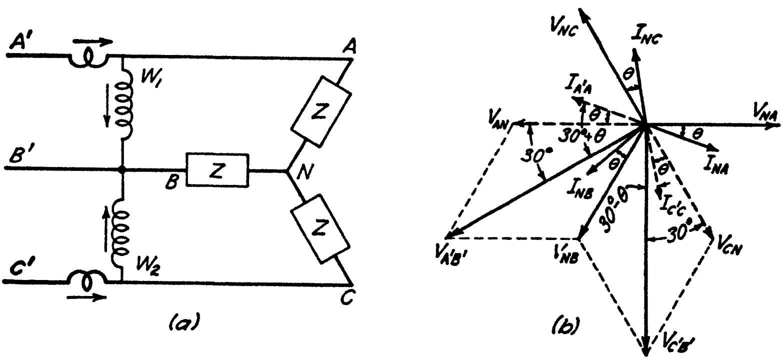


FIG. 5–13. Connection diagram and vector diagram for measuring three-phase power with two wattmeters when the load is balanced.

wye-connected load shown in Fig. 5–13 which presupposes that the meters in the diagram are so connected that the currents in the potential coil and current coil will cause both meters to read correctly. The arrows on the diagram show the current direction in the two coils, and

the subscripts must be read in that direction in the same way that they are read for power determination from complex quantities (Art. 14, Chapter 11).

Wattmeter W_1 is affected by voltage $V_{A'B'}$ and current $I_{A'A}$, while wattmeter W_2 is affected by voltage $V_{C'B'}$ and current $I_{C'C}$. In each instance, there will also be a displacement angle between the voltage and current. Assume that the three voltages measured to neutral are V_{NA}, V_{NB}, and V_{NC}. Then, by Kirchhoff's Law,

$$\overline{V}_{A'B'} = \overline{V}_{NB} + \overline{V}_{AN}$$

$$\overline{V}_{C'B'} = \overline{V}_{NB} + \overline{V}_{CN}$$

$$\overline{I}_{A'A} = \overline{I}_{AN}$$

$$\overline{I}_{C'C} = \overline{I}_{CN}$$

By construction, the two voltages $V_{A'B'}$ and $V_{C'B'}$ can be placed on Fig. 5–13*b* and the resultant angular relationship between voltage and current vectors in the wattmeters is shown to be $\cos(30° + \theta)$ and $\cos(30° - \theta)$ when θ is the power factor angle of the load. The power read by each of the wattmeters is

$$W_1 = V_L I_L \cos(30° + \theta) \qquad (b\text{–}13)$$

$$W_2 = V_L I_L \cos(30° - \theta) \qquad (c\text{–}13)$$

where V_L and I_L are the line voltage and current.

Expanding and adding, the sum of the two wattmeter readings $(W_1 + W_2)$ can be shown to be equal to the total power in a balanced system.

$$W_1 = V_L I_L(\cos\theta\cos 30° - \sin\theta\sin 30°)$$

$$W_2 = V_L I_L(\cos\theta\cos 30° + \sin\theta\sin 30°)$$

$$W_1 + W_2 = V_L I_L(2\cos\theta\cos 30°)$$

$$= V_L I_L\left(2\cos\theta\frac{\sqrt{3}}{2}\right)$$

$$= \sqrt{3}V_L I_L\cos\theta$$

The power factor will influence the reading of the two meters and, when the power factor is unity,

$$W_1 = V_L I_L \cos 30°$$

$$W_2 = V_L I_L \cos 30°$$

and the two meters will read the same.

If the power-factor angle is 60°, the two meter readings are,

$$W_1 = 0$$

$$W_2 = V_L I_L \cos 30°$$

If the power-factor angle exceeds 60°, the meters will have different signs and, numerically, will subtract instead of add, because an angle greater than 90° has a negative cosine. Since it is necessary to connect the meters so that they both read up-scale, some additional knowledge is necessary to determine whether the meter readings are to be added or subtracted. That knowledge is available when the character of the load is definitely known, but such information is rarely available.

It is possible to combine the meter readings in such a manner that the value of the power-factor angle can be determined. Consider equations (b–13) and (c–13) for wattmeter readings W_1 and W_2. By addition and subtraction, the following is obtained.

$$W_2 + W_1 = 2V_L I_L \cos\theta \cos 30°$$

$$W_2 - W_1 = 2V_L I_L \sin\theta \sin 30°$$

and

$$\frac{W_2 - W_1}{W_2 + W_1} = \frac{V_L I_L}{\sqrt{3} V_L I_L} \times \frac{\sin\theta}{\cos\theta}$$

Expressed in the form of the tangent of the angle, this gives

$$\tan\theta = \sqrt{3}\,\frac{W_2 - W_1}{W_2 + W_1} = \sqrt{3}\,\frac{\text{difference of readings}}{\text{sum of readings}} \qquad (d\text{–}13)$$

That is, the tangent of the power-factor angle is the product of 1.732 and the ratio of the difference and sum of the two wattmeter readings. Figure 6–13 shows a graph for determining the power factor from meter reading ratios. This method is easier than the solution of equation (d–13) and these charts, in various forms, are often found in the covers of wattmeter cases.

This chart is used for determining power in *balanced,* three-phase systems.

Example a. Two wattmeters measuring the power in a balanced three-phase system read 13.6 and 34.3 kw, respectively. If the line current is 20 amp and the load is connected in wye, what will be the impedance of each phase load expressed in complex notation? Assume the current lagging.

$$\tan\theta = \sqrt{3}\,\frac{20.7}{47.9} = 0.75$$

$$\theta = 36.9°$$

$$\cos\theta = 0.8$$

$$\sin\theta = 0.6$$

$$R = \frac{W}{3I^2} = \frac{47{,}900}{3 \times (20)^2} = 40 \text{ ohms}$$

$$X = 0.75 \times 40 = 30 \text{ ohms}$$

The load has a lagging current, and

$$\bar{Z} = 40 + j30$$

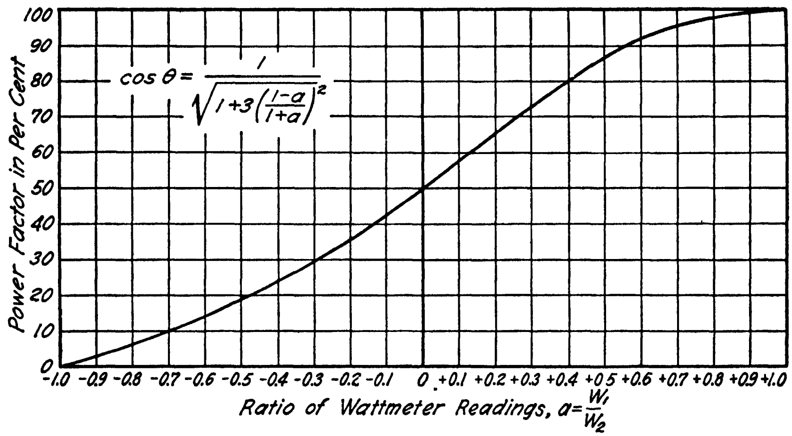


FIG. 6–13. Chart for determining the power factor of a balanced three-phase system from two wattmeter readings. The sign of the meter readings must be known.

4. Laboratory Method of Determining the Sign of Two Wattmeter Readings Measuring Three-Phase Power. It has been shown, by use of the tangent of the power-factor angle, how to determine whether the power factor of a balanced three-phase system is greater or less than 0.5 when the arithmetical sum or difference of the meters is taken and the sign of the meter reading is known.

The nature of the sign may be determined by replacing one of the wattmeters with the other without disturbing the relative connection of the current and potential coils. Figure 7–13*a* shows the connection of two wattmeters, using the polarity signs marked on the meter. Both meters should read up-scale with the reversing switch (if there is a reversing switch) in the normal position. This is a positive reading for both meters. If the terminals must be reversed, as in Fig. 7–13*b*, for wattmeter W_2, that meter has a negative reading and the total power is the difference of the two meters.

Again, in Fig. 7–13*b*, to move W_1 to the position W_2 requires that the leads be reversed to make the meter read up-scale, and indicates

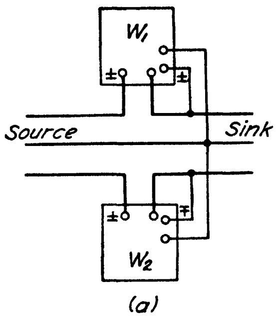


FIG. 7–13. Relative connection of wattmeters for determining the sign of the meter readings.

that the difference of the two meter readings will give the total power. If, as in Fig. 7–13*b*, the change of W_1 to the position W_2 does not require the leads to be reversed, the sum of the two meter readings is the total power.

5. General Proof that Two Wattmeters Will Indicate Any Three-Phase Three-Wire Power. The use of complex quantities simplifies the general demonstration that the three-wire three-phase system, when *balanced* or *unbalanced*, may be measured with two wattmeters and also demonstrates some of the advantages of the use of the symbolic method.

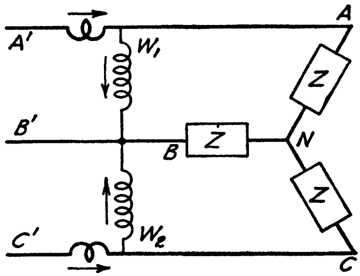


FIG. 8–13. Proper diagram and meter connections of two wattmeters for determining the power from the complex expressions for voltage and current. The arrows on the voltage and current coils indicate the direction in which the voltage and current should be taken.

Figure 8–13 shows a wye load connected to a power source, with two wattmeters W_1 and W_2 connected to read the total power of the system. If the two meters deflect in the correct direction, the following currents and voltages are used.

W_1	W_2
$\bar{V}_{AB} = v_{AB} + jv_{AB}'$	$\bar{V}_{CB} = v_{CB} + jv_{CB}'$
$\bar{I}_{AN} = i_{AN} + ji_{AN}'$	$\bar{I}_{CN} = i_{CN} + ji_{CN}'$

If the indication of each meter is to be determined from the complex expressions of V and I,

$$W_1 = \overline{\bar{V}_{AB} \mid \bar{I}_{AN}} = v_{AB}i_{AN} + v_{AB}'i_{AN}'$$

$$W_2 = \overline{\bar{V}_{CB} \mid \bar{I}_{CN}} = v_{CB}i_{CN} + v_{CB}'i_{CN}'$$

and the total power (the sum of the two wattmeter readings) is

$$W_t = W_1 + W_2 = v_{AB}i_{AN} + v_{CB}i_{CN} + v_{AB}'i_{AN}' + v_{CB}'i_{CN}' \qquad (e\text{–}13)$$

It is necessary to eliminate the line voltages and to introduce the phase voltages in order to prove that the total power in the phases is equal to the power read by the two wattmeters. By Kirchhoff's Law,

$$\begin{aligned}\bar{V}_{AB} &= \bar{V}_{AN} + \bar{V}_{NB}\\ &= v_{AN} + jv_{AN}' + v_{NB} + jv_{NB}'\\ &= (v_{AN} + v_{NB}) + j(v_{AN}' + v_{NB}')\end{aligned}$$

By substitution, the indication on the first wattmeter is

$$W_1 = (v_{AN} + v_{NB})i_{AN} + (v_{AN}' + v_{NB}')i_{AN}'$$

In the same way, the indication on the second wattmeter is

$$W_2 = (v_{CN} + v_{NB})i_{CN} + (v_{CN}' + v_{NB}')i_{CN}'$$

If these values for W_1 and W_2 are used, equation (e–13) becomes

$$\begin{aligned}W_1 + W_2 = v_{AN}i_{AN} + v_{NB}i_{AN} + v_{CN}i_{CN} + v_{NB}i_{CN} + v_{AN}'i_{AN}'\\ + v_{NB}'i_{AN}' + v_{CN}'i_{CN}' + v_{NB}'i_{CN}'\end{aligned}$$

$$\begin{aligned}W_1 + W_2 = v_{AN}i_{AN} + v_{CN}i_{CN} + v_{NB}(i_{AN} + i_{CN}) + v_{AN}'i_{AN}'\\ + v_{CN}'i_{CN}' + v_{NB}'(i_{AN}' + i_{CN}') \qquad (f\text{–}13)\end{aligned}$$

By Kirchhoff's Law,

$$\bar{I}_{AN} + \bar{I}_{CN} + \bar{I}_{BN} = 0$$

$$i_{AN} + ji_{AN}' + i_{CN} + ji_{CN}' + i_{BN} + ji_{BN}' = 0$$

When a complex equation is equated to zero, the sum of the real components and the sum of the imaginary components are each equal to zero; therefore,

$$i_{AN} + i_{CN} + i_{BN} = 0$$

$$i_{AN}' + i_{CN}' + i_{BN}' = 0$$

and

$$i_{AN} + i_{CN} = i_{NB}$$

$$i_{AN}' + i_{CN}' = i_{NB}'$$

Substituting these values of i_{NB} and i_{NB}' in equation (f–13),

$$
\begin{aligned}
W_1 + W_2 &= v_{AN}i_{AN} + v_{CN}i_{CN} + v_{NB}i_{NB} + v_{AN}'i_{AN}' + v_{CN}'i_{CN}' \\
&\qquad + v_{NB}'i_{NB}' \\
&= (v_{AN}i_{AN} + v_{AN}'i_{AN}') + (v_{CN}i_{CN} + v_{CN}'i_{CN}') \\
&\qquad + (v_{NB}i_{NB} + v_{NB}'i_{NB}' \qquad (g\text{–}13)
\end{aligned}
$$

The reversal of the subscript $v_{NB}i_{NB}$ to $v_{BN}i_{BN}$ follows the law of change of subscript and sign; therefore,

$$v_{NB}i_{NB} = (-v_{BN})(-i_{BN}) = v_{BN}i_{BN}$$

The power in the three loads composing the three-phase load will be

$$
\begin{aligned}
P_{AN} &= \overline{\bar{V}_{AN} \mid \bar{I}_{AN}} & \qquad P_{AN} &= v_{AN}i_{AN} + v_{AN}'i_{AN}' \\
P_{BN} &= \overline{\bar{V}_{BN} \mid \bar{I}_{BN}} & \qquad P_{BN} &= v_{BN}i_{BN} + v_{BN}'i_{BN}' \\
P_{CN} &= \overline{\bar{V}_{CN} \mid \bar{I}_{CN}} & \qquad P_{CN} &= v_{CN}i_{CN} + v_{CN}'i_{CN}'
\end{aligned}
$$

Substituting the power symbols in equation (g–13),

$$W_1 + W_2 = P_{AN} + P_{BN} + P_{CN}$$

or, the two wattmeters will indicate the total power in the three-phase circuit. Since the development is by the use of the complex values for the current and voltage, the impedances may be of any value and there are no limitations on the system. This forms a general proof for any type of load or combinations of loading.

Example b. Two wattmeters connected to measure the power in a three-wire three-phase system have the following complex currents and voltages impressed upon their windings: W_1 current coil $\bar{I}_{AN} = 2.5 - j9$, potential coil $\bar{V}_{AB} = 100 + j0$; W_2 current coil $\bar{I}_{CN} = -1.6 + j10$, and potential coil $\bar{V}_{CB} = 50 + j86.6$. What is the power read by each instrument, and what is the power consumed by the system?

$$
\begin{aligned}
W_1 &= \overline{\bar{V}_{AB} \mid \bar{I}_{AN}} \\
&= (2.5 \times 100) + (-9 \times 0) \\
&= 250 \text{ watts} \\
W_2 &= \overline{\bar{V}_{CB} \mid \bar{I}_{CN}} \\
&= (-1.6 \times 50) + (10 \times 86.6) \\
&= 786 \text{ watts}
\end{aligned}
$$

$$P = W_1 + W_2 = 250 + 786 = 1036 \text{ watts}$$

6. Reactive Volt-Amperes in a Three-Phase Circuit. Single-phase reactive volt-amperes are expressed by

$$Q_p = V_p I_p \sin \theta_p$$

which may be determined from the single-phase loads when the voltage, current, and impedance of the individual loads are known. If this is referred to line conditions,

$$Q_p = V_L \frac{I_L}{\sqrt{3}} \quad \sin \theta_p \text{ for delta systems}$$

and

$$Q_p = \frac{V_L}{\sqrt{3}} I_L \quad \sin \theta_p \text{ for wye systems}$$

The total reactive volt-amperes in the balanced three-phase system will be

$$Q = \frac{3}{\sqrt{3}} V_L I_L \sin \theta_p$$

In either the delta or wye system, therefore, the reactive volt-amperes in a three-phase balanced system will be

$$Q = \sqrt{3} V_L I_L \sin \theta_p$$

It has been shown for the single-phase circuit that

$$\cos \theta = \frac{P}{\sqrt{P^2 + Q^2}}$$

and, since power in the polyphase system is a scalar quantity, the $\cos \theta$ (not an angle between V_L and I_L) is expressed by

$$\cos \theta = \frac{\Sigma P}{\sqrt{\Sigma P^2 + \Sigma Q^2}} \qquad (h\text{–}13)$$

This, by definition, would indicate that the hypothetical power factor of an unbalanced three-phase system is the power factor of an equivalent balanced circuit having the same total active and reactive components. Applied to the balanced three-phase circuit, the equation becomes

$$\cos \theta = \frac{3P}{3\sqrt{P^2 + Q^2}} = \frac{P}{\sqrt{P^2 + Q^2}}$$

which is the same as for a single-phase circuit. The power factor of the balanced three-phase circuit is the power factor of one phase of the circuit.

7. Measurement of Reactive Volt-Amperes in a Balanced Three-Phase System with One Meter. If a three-phase system is balanced, the reactive volt-amperes are measured as shown in Fig. 9–13, when the voltage and current effecting the deflection will be $V_{C'A'}$ and I_{BN}. The voltage, by Kirchhoff's Law, will be

$$\overline{V}_{C'A'} = \overline{V}_{CN} + \overline{V}_{NA}$$

which is indicated on vector diagram, Fig. 9–13. The angle between V_{CA} and I_{BN} is $(90° - \theta_p)$, and the reading of the meter will be

$$W_Q = VI \cos (90° - \theta_p)$$

$$W_Q = VI \sin \theta_p$$

the reactive volt-amperes of a single meter must be corrected to read properly for the three-phase system.

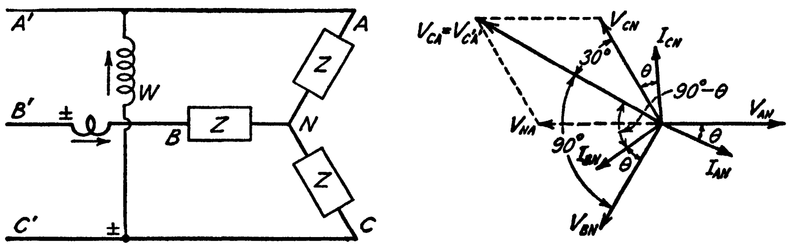


FIG. 9–13. The connection diagram and vector diagram when using one wattmeter to determine the reactive volt-amperes in a balanced three-phase system.

It has been shown that the reactive volt-amperes, of a balanced three-phase system, are expressed by

$$Q = 3V_pI_p \sin \theta_p \qquad (i\text{–}13)$$

The meter as read in Fig. 9–13, will give an indication with respect to the line current and voltage so that

$$W_Q = V_LI_L \cos (90° - \theta_p)$$
$$= \sqrt{3}V_pI_p \sin \theta_p$$

If equation (i–13) is changed to fit the line conditions for a wye system,

$$Q = 3\frac{V_L}{\sqrt{3}} I_L \sin \theta_p$$

or

$$Q = \sqrt{3}V_LI_L \sin \theta_p$$

It is only necessary to multiply the meter reading by $\sqrt{3}$ to obtain the reactive volt-amperes for a balanced three-phase system. Therefore,

$$Q = \sqrt{3}W_Q$$

Example c. A balanced three-phase impedance, $\bar{Z} = 4 + j3$, is connected in wye across a balanced three-phase voltage of 173.2 volts. What will be the total reactive power determined by a wattmeter connected as shown in Fig. 9–13?

$$V = 173.2 \text{ volts}$$

$$V_p = \frac{173.2}{\sqrt{3}} = 100 \text{ volts}$$

$$I = I_p = \frac{100}{\sqrt{4^2 + 3^2}} = 20 \text{ amp}$$

$$\cos\theta = \frac{4}{\sqrt{4^2 + 3^2}} = 0.8$$

$$\sin\theta = 0.6$$

$$Q = \sqrt{3} \times 173.2 \times 20 \times 0.6 = 3600 \text{ vars}$$

8. Measurement of Three-Phase Reactive Volt-Amperes, Using Reactive Compensators. In Chapter 11, it was shown how reactive volt-amperes entered into the fixing of rates because of the additional losses and fixed charges on a system. These losses and fixed charges are caused by a poor power factor, produced by large inductances and by induction motors operating below rated capacity.

The important problem lies with the utilities, for, in the transfer of large blocks of power from one network to another, the character of that power is of prime importance. The excitation on generators of one generating station can be so arranged that the undesirable reactive component is shifted to a neighboring station and, since no company wishes to carry unnecessary reactive volt-amperes, the utilities guard their interests and rights by elaborate metering and relay installations.

The reactive volt-amperes in a balanced three-phase system are no more complicated than those of a single-phase system and are definitely established (except for the algebraic sign). Since the reactive volt-amperes may be either positive or negative, it is necessary to establish whether capacity or inductance causes positive reactive volt-amperes. In the previous discussions of reactive volt-amperes, it was decided to consider inductive load negative.

Figure 10–13 shows the circuit diagram of meters connected to a system by means of a reactive compensator and the vector diagram for the connection when the reactive volt-amperes of a balanced three-phase system having lagging current are being measured.

9. Measurement of Reactive Three-Phase Volt-Amperes in a Balanced-Voltage Three-Phase System, Using Two Wattmeters and a Reactive Compensator. Though the foregoing method * can be used for measuring unbalanced systems which have balanced voltages, the discussion will be confined to the balanced system. The study of the unbalanced system problem is complicated and there are many moot questions regarding definitions and interpretation.

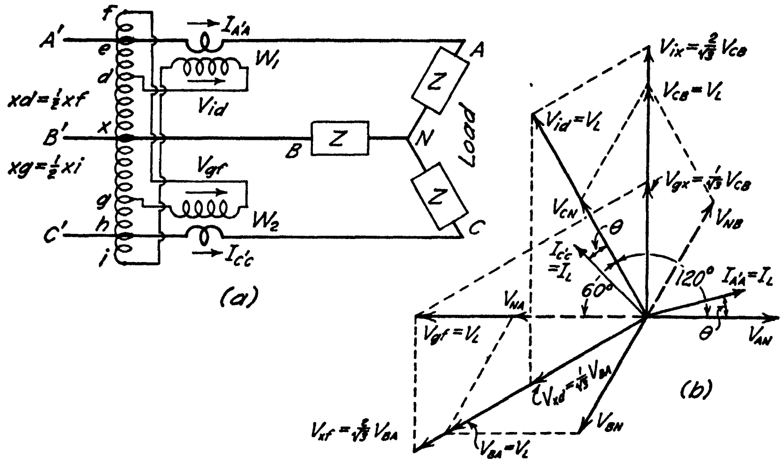


FIG. 10–13. Circuit and vector diagram showing the connection of two wattmeters with reactive compensators to read the reactive volt-amperes of a balanced three-phase system, arranged to make a leading current read positive.

When using "phasing transformers" or "reactive compensators," the above system of autotransformers is connected in open delta and is frequently used for dual service: shifting the voltage 90° ahead to make leading current (capacitive load) positive and 90° behind to make lagging current (inductive load) negative in phase position and adjusting the line voltage to the rated voltage of the meter coils in the potential circuits. By extending the windings of the autotransformers, the necessary 15.5 per cent, used for making the meters direct reading, is obtained. This correction must be made numerically when using a single meter for measuring the balanced three-phase load. The autotransformer has the output winding tapped at the midpoint.

* For a comprehensive discussion of reactive metering and the errors involved, reference may be made to Knowlton's *Electric Power Metering*, McGraw-Hill, 1934, or to the report of the Sub-committee of the Engineering Section, Great Lakes Division, NELA, April 23, 1930, A. R. Knight, Chairman.

In Fig. 10–13, starting with the voltages in the individual phases, the diagram may be constructed to determine the voltages $\overline{V}_{gf}$ and $\overline{V}_{id}$, which are the voltages on the potential coils of the wattmeters. The line currents through the current coils are the phase currents. The voltages on the potential coils will be equal to the line voltages, and this makes it unnecessary to multiply the results by the factor $\sqrt{3}$, this correction having been made by the reactive compensator.

The meters are connected in Fig. 10–13*b* to give a positive reading for capacitive load as required by the definitions of the A.S.A. as discussed in Chapter 11. The choice of conventions in mathematics and in electrical engineering has nothing to do with the natural phenomena. In practice the wattmeters are so connected that the inductive load gives a positive reading, regardless of the conventions recommended by the definitions.

When the "unit" system is used in making an analysis of the vector diagram for the installation, the phase voltages are given by

$$\overline{V}_{AN} = 1 + j0, \quad V_{AN} = 1$$

$$\overline{V}_{BN} = -\frac{1}{2} - j\frac{\sqrt{3}}{2}, \quad V_{BN} = 1$$

$$\overline{V}_{CN} = -\frac{1}{2} + j\frac{\sqrt{3}}{2}, \quad V_{CN} = 1$$

To determine the voltage V_{gf}, substitute in

$$\begin{aligned}
\overline{V}_{gf} &= \overline{V}_{gx} + \overline{V}_{xf} \\
&= \frac{1}{\sqrt{3}}\overline{V}_{CB} + \frac{2}{\sqrt{3}}\overline{V}_{BA} \\
&= \frac{1}{\sqrt{3}}(\overline{V}_{CN} + \overline{V}_{NB}) = \frac{2}{\sqrt{3}}(\overline{V}_{BN} + \overline{V}_{NA}) \\
&= \frac{1}{\sqrt{3}}\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2} + \frac{1}{2} + j\frac{\sqrt{3}}{2}\right) \\
&\qquad + \frac{2}{\sqrt{3}}\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2} - 1 + j0\right) \\
&= -1.732 + j0 = |\,1.732\,| \text{ line voltage}
\end{aligned}$$

To determine the voltage V_{id}, substitute in

$$\begin{aligned}
\overline{V}_{id} &= \overline{V}_{ix} + \overline{V}_{xd} \\
&= \frac{2}{\sqrt{3}}\overline{V}_{CB} + \frac{1}{\sqrt{3}}\overline{V}_{BA} \\
&= \frac{2}{\sqrt{3}}(\overline{V}_{CN} + \overline{V}_{NB}) + \frac{1}{\sqrt{3}}(\overline{V}_{BN} + \overline{V}_{NA}) \\
&= \frac{2}{\sqrt{3}}\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2} + \frac{1}{2} + j\frac{\sqrt{3}}{2}\right) \\
&\quad + \frac{1}{\sqrt{3}}\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2} - 1 + j0\right) \\
&= -0.866 + j1.5 = |1.732| \text{ line voltage}
\end{aligned}$$

For leading current (as shown in the diagram),

$$W_{Q_1} = V_L I_L \cos(120° - \theta)$$

$$W_{Q_2} = V_L I_L \cos(60° - \theta)$$

$$W_{Q_1} = V_L I_L\left(-\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right)$$

$$W_{Q_2} = V_L I_L\left(\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right)$$

$$\begin{aligned}
Q = W_{Q_1} + W_{Q_2} &= V_L I_L\left(2 \times \frac{\sqrt{3}}{2}\sin\theta\right) \\
&= \sqrt{3}\, V_L I_L \sin\theta \qquad (j\text{–}13)
\end{aligned}$$

For lagging current when the angle θ is negative instead of positive with respect to the phase voltage,

$$W_{Q_1} = V_L I_L \cos(120° + \theta)$$

$$W_{Q_1} = V_L I_L \cos(60° + \theta)$$

$$Q = W_{Q_2} + W_{Q_2} = -\sqrt{3}\, V_L I_L \sin\theta \qquad (k\text{–}13)$$

The readings of the two wattmeters, where V and I are the line voltage and current respectively, as given in equations (j–13) and (k–13) are the reactive volt-amperes in a balanced three-phase system in each instance. The signs conform to the conventions established by definition.

Example d. A three-phase balanced inductive load of 86.6 per cent power factor is connected to a three-phase balanced voltage of 100 volts. What will be the reading of the two wattmeters (connected to read reactive volt-amperes) using reactive compensators, if the line current is 10 amp?

$$
\begin{aligned}
W_{Q_1} &= VI \cos (60° + \theta) \\
&= 100 \times 10 \times \cos (60° + 30°) \\
&= 0 \text{ vars} \\
W_{Q_2} &= VI \cos (120° + \theta) \\
&= 100 \times 10 \times \cos (120° + 30°) \\
&= -866 \text{ vars} \\
W_{Q_1} + W_{Q_2} &= Q = 0 - 866 = -866 \text{ vars} \quad \text{(negative for inductive load)} \\
Q &= \sqrt{3} VI \sin \theta \\
&= \sqrt{3} \times 100 \times 10 \times (-0.5) \quad (\text{with } V \text{ as a reference, } \theta = -30° \text{ and } \sin \theta \text{ is negative}) \\
&= -866 \text{ vars} \quad \text{(negative for inductive load)}
\end{aligned}
$$

The foregoing is given for the purpose of demonstrating the value of the use of complex notation and the analysis of electrical engineering problems by means of "unit" system and vector diagrams.

10. Summary. The following are the various forms for three-phase power.

Active Power (P in watts)	Reactive Power (Q in vars)	Apparent Power (P_A in volt-amperes)
$P = \sqrt{3} VI \cos \theta$	$Q = \sqrt{3} VI \sin \theta$	$P_A = \sqrt{P^2 + Q^2}$

In three-phase systems the power can be obtained with two wattmeters as follows.

Balanced systems:

$$
\begin{aligned}
P &= W_1 + W_2 & Q &= Q_1 + Q_2 \\
W_1 &= VI \cos (30° + \theta) & Q_1 &= VI \sin (\theta + 30°) \\
W_2 &= VI \cos (30° - \theta) & Q_2 &= VI \sin (\theta - 30°) \\
\tan \theta &= \sqrt{3} \frac{W_1 - W_2}{W_1 + W_2}
\end{aligned}
$$

Any three-wire three-phase system:

$$
\begin{aligned}
P &= W_1 + W_2 \\
&= \overline{\bar{V}_1 \,|\, \bar{I}_1} + \overline{\bar{V}_2 \,|\, \bar{I}_2} \\
&= v_1 i_1 + v_1' i_1' + v_2 i_2 + v_2' i_2' \\
Q &= Q_1 + Q_2 \\
&= \underline{\bar{V}_1 \,|\, \bar{I}_1} + \underline{\bar{V}_2 \,|\, \bar{I}_2} \\
&= v_1 i_1' - v_1' i_1 + v_2 i_2' - v_2' i_2
\end{aligned}
$$

PROBLEMS

(Circuit designations are those used with the figures in the chapter.)

1–13. Two wattmeters on a 3-phase, "Y" 100-volt line measure the power used by a balanced load where $\bar{Z} = 8.66 + j5$. (*a*) What will the two wattmeters read? (*b*) What is the apparent power, (*c*) the active power, (*d*) the reactive power?

2–13. A delta-connected load, with 5 ohms in each phase, is connected to a 3-phase, 100-volt line. The power factors of the three loads are 60, 80 lagging, and 100 per cent respectively. What is (*a*) the power consumed in each phase and (*b*) the total power?

3–13. A balanced 3-phase voltage of 100 volts is connected to three wires AA', BB', CC' and each wire carries 20 amp to a load with an 86.6 per cent lagging power factor. Determine the reading of wattmeters connected to read the load.

4–13. A balanced "Y" load, $\bar{Z} = 4 + j3$, is connected to a 173.2-volt, 3-phase supply; two wattmeters are connected to indicate the power and another is used for determining the reactive power. Show how the meters are connected and determine the reading of each meter.

5–13. A balanced 3-phase, "Δ" load $\bar{Z} = 3 + j4$ is connected to a 3-phase, 200-volt line. Determine (*a*) the reactive, (*b*) the active, (*c*) the apparent power, and (*d*) the reading of two wattmeters connected to read the total load.

6–13. On a 3-wire, 1-phase system the voltages $\bar{V}_{BA} = 80 + j60$ and $\bar{V}_{CB} = 80 + j60$ are impressed from the lines to neutral. Impedances are connected with $\bar{Z}_1 = 8 + j6$ across the line $A'B'$ and two impedances in parallel ($\bar{Z}_2 = 16 + j12$, $\bar{Z}_3 = 16 + j12$) across $B'C'$. Determine the indication on a wattmeter with the current coil in line AA' and the potential coil across AB.

7–13. When two wattmeters are used to measure a balanced 3-phase, wye-connected load the readings are 137 and 1063 watts with 11.55 amp at a lagging power factor flowing in the lines. Determine (*a*) the complex value of the impedance and (*b*) the power factor of the system.

8–13. For a 3-phase delta-connected load the voltages and impedances are

$$
\begin{aligned}
\bar{V}_{AB} &= 50 + j86.6 & \qquad \bar{Z}_{AB} &= 8.66 - j5 \\
\bar{V}_{BC} &= -100 + j0 & \qquad \bar{Z}_{BC} &= 7.07 + j7.07 \\
\bar{V}_{CA} &= 50 - j86.6 & \qquad \bar{Z}_{CA} &= 8.66 + j5
\end{aligned}
$$

What will be (*a*) the active and (*b*) the reactive power?

9–13. An unbalanced delta load, $\bar{Z}_{AB} = 0 - j10$, $\bar{Z}_{BC} = 10 + j0$, and $\bar{Z}_{CA} = 0 + j10$, is connected to a 3-phase, 100-volt system with $\bar{V}_{CB} = 100 + j0$ and a phase sequence ABC. The current coil of wattmeter 1 is in line $A'A$ and that of wattmeter 2 is in line $B'B$ with the common potential connection on line CC'. What wattage will each meter indicate?

10–13. A 3-phase, 4-wire, 100-volt to neutral system has the following voltages and impedances

$$\bar{V}_{AN} = 100 + j0 \qquad \bar{Z}_{AN} = 3 + j4$$

$$\bar{V}_{BN} = -50 + j86.6 \qquad \bar{Z}_{BN} = 5 + j0$$

$$\bar{V}_{CN} = -50 - j86.6 \qquad \bar{Z}_{CN} = 5 - j8.66$$

A wattmeter is connected with the current coil in the neutral and the potential coil across AN. What is the wattmeter reading?

11–13. A load is connected in open delta across a 3-phase, 100-volt system. The voltage $\bar{V}_{AB} = 100 + j0$ and the sequence is ABC. The load impedances are $\bar{Z}_{BC'} = 10 + j0$ and $\bar{Z}_{AC'} = 0 + j10$. The current coil of W_1 is in line AA', the potential coil across AB, and the current coil of W_2 is in line CC', the potential coil across BC. What is (*a*) the reading of each meter, (*b*) the total power consumed, (*c*) the total reactive power?

12–13. In a 3-phase, 4-wire, 173.2-volt system, $\bar{V}_{AN} = 100 + j0$ and the sequence is ABC. The connected impedances are $\bar{Z}_{AN} = 8 + j6$, $\bar{Z}_{BN} = 10 + j0$, and $\bar{Z}_{CN} = 10 + j0$. With wattmeters in each line and the potential coils connected to neutral determine (*a*) the readings on the individual meters and (*b*) the total power.

13–13. An unbalanced delta load is connected to a 3-phase, 200-volt system where $\bar{V}_{AB} = 200 + j0$ and the phase sequence is ABC. The impedances Z_1, Z_2, and Z_3 in the phases $A'B'$, $B'C'$, and $C'A'$ take 1732 watts at 86.6 per cent leading power factor, 1000 watts at 50 per cent lagging power factor, and 2000 watts at unity power factor respectively. What will the wattmeters read if the current coils are placed in lines $A'A$ and $B'B$?

14–13. A delta load of 5 ohms impedance and 80 per cent lagging power factor is connected to a 3-phase, 100-volt system. Without the use of complex quantities determine (*a*) the active, (*b*) the reactive, (*c*) the apparent power, and (*d*) the two wattmeter readings, using complex quantities.

15–13. A 3-phase, 4-wire system has a line voltage of 173.2 volts, with $\bar{V}_{AB} = 173.2 + j0$ and the sequence is ABC. The wye-connected load is $\bar{Z}_{C'N'} = 4 + j3$, $\bar{Z}_{B'N'} = 3 + j4$, $\bar{Z}_{A'N'} = 5 + j0$ with the wattmeter coils connected in lines CC', AA', and NN' for wattmeters W_1, W_2, and W_3 respectively. Determine from complex quantities (*a*) the readings of the three wattmeters, (*b*) the reactive power from the meter indications. and (*c*) the total volt-amperes.

CHAPTER 14

POWER DISTRIBUTION

1. Generation, Transmission, and Distribution of Power. Electrical power supply may be divided into three parts: (1) generation, (2) transmission, and (3) distribution. With the exception of electric transportation, the major part of the electrical power used is supplied by a-c systems. The approximate voltage limits for the three parts, generation, transmission, and distribution, are

	A-C Systems	D-C Systems
Generation	2,300 to 22,000 volts	110 to 1500 volts
Transmission	33,000 to 287,000 volts	110 to 1500 volts
Distribution	110 to 6,600 volts	110 to 1500 volts

The low voltage of d-c systems limits its use to areas where the load density is high. If power is to be transmitted long distances, the current per conductor must be small in order that the line loss be low and the efficiency high. For this reason, alternating current is used whenever large amounts of power are to be transmitted. The high voltages of transmission systems are obtained by using transformers to increase the generated voltage at the generating end and to reduce the voltage at the receiving end.

Voltages for lighting and small power loads are 110 to 220 volts for both a-c and d-c systems but, for large power requirements, the voltages are 440 and 2300 volts, three-phase alternating current, and 550 volts direct current.

2. Generation. The power required by the electrical systems of most operating companies is supplied by several generators operating in parallel. The electrical rating of a generator determines its output capacity and the electrical load of the system must be divided among the generators so that the ratings are not exceeded. Single a-c generators with capacities of 165,000 and 105,000 kilovolt-amperes have been built and one generating unit (State Line Station) with a capacity of 208,000 kilovolt-amperes is in operation. Generator capacities selected for modern installations are usually either 25,000, 50,000, or 100,000 kilovolt-amperes.

Generator voltages above 25,000 volts are not advisable at the present time, because it is difficult to insulate the conductors of the armature windings and the terminal connections.

3. Transmission. Resistance is the only opposition offered by the conductors in the transmission of power by direct current but, in an a-c system, the oppositions are both resistance and reactance (inductive and capacitive).

The power loss in a transmission line must be considered in terms of the power transferred. High voltage systems are used in order to reduce the size of the conductor and current per conductor and, consequently, reduce the line loss. Table I–14 shows the requirements in copper as the voltage of a system is increased from 120 to 120,000 volts, the distance, load, and per cent voltage drop being constant.

TABLE I–14

COPPER REQUIREMENTS AT VARIOUS VOLTAGES

Using a two-wire system with per cent drop, load, and distance constant.

Voltage	Copper (Per Cent)
120	100
240	25
600	4
1,200	1
2,400	0.25
12,000	0.01
24,000	0.0025
120,000	0.0001

(*a*) *Resistance.* The resistance of a conductor depends upon its material, size, and length. Copper is the most common material for conductors, especially the smaller sizes. Rural lines often use a copper-coated steel conductor which has increased resistance per unit length because of the steel. Some lines have aluminum conductors cabled around a steel core (for strength) and the resistance of both the wire and core must be considered in the line resistance calculations. The transmission lines from Boulder Dam are large hollow-cored copper conductors. This gives a large outside surface to the conductor (an important factor in reducing corona loss) and, at the same time, keeps the weight to a minimum.

(*b*) *Reactance.* The inductive and capacitive reactances of transmission lines depend upon the line material and construction. The spac-

ing distance, which is dependent upon the voltage, affects both the inductance and capacity of the line.

A conductor carrying a current is surrounded by a magnetic field which is produced by the current. This magnetic field links the conductor, which forms a single loop, and the change in flux linkages resulting from the alternating current produces inductive reactance. If the distance of transmission is short, the effect of inductance is usually small, but it is advisable to calculate the reactance to determine if it can be omitted. The inductance per mile for a single conductor can be determined from the expression

$$\mathcal{L}_{\text{per mile}} = \left(0.0805 \frac{\mu}{\mu_a} + 0.74 \log_{10} \frac{D}{R}\right) 10^{-3} \text{ henry}$$

The inductive reactance per mile can be found if this expression is multiplied by the constant $2\pi f$ and

$$X_{\mathcal{L}_{\text{per mile}}} = 2\pi f \left(0.0805 \frac{\mu}{\mu_a} + 0.74 \log_{10} \frac{D}{R}\right) 10^{-3} \text{ ohm}$$

D = equivalent distance between conductors in inches

R = radius of the conductor in inches

μ = permeability of conductor material

μ_a = permeability of air

This expression gives the value of inductive reactance per conductor per mile and is useful in determining the values for single-phase and three-phase systems. It will be noted that two terms compose the inductance term of the reactance expression. One of the terms depends upon the size and spacing of the conductors, and the other term depends only on the permeability. Usually, in overhead line construction, the first term of the expression is much smaller than the second, unless iron or steel conductors are used. When the conductors consist of copper or aluminum the value of μ/μ_a is 1. If the above expression is used for three-phase systems, the conductors should be spaced symmetrically and form an equilateral triangle. However, if the conductor spacing is not symmetrical and conductors are transposed so that equal exposure in all conductor positions is obtained, it is possible to determine the equivalent spacing for the conductors. If D_1, D_2, and D_3 are the spacings between the conductors the equivalent spacing can be determined from

$$D = \sqrt[3]{D_1 D_2 D_3} \qquad \text{equivalent spacing}$$

In addition to the inductance of a conductor, the capacitance between conductors must be considered, since parallel insulated conductors with an air dielectric between them form a condenser. The capacitance is important in high voltage cables, where the conductors are close together. The capacitance of long open-wire, high voltage lines will require an appreciable charging current. The capacitance between two parallel conductors is

$$C_{\text{per mile}} = \frac{19.41 \times 10^{-9}}{\log_{10} \frac{D}{R}} \text{ farads}$$

and the capacitance of a single conductor to neutral (ground) is

$$C_{\text{per mile}} = \frac{38.83 \times 10^{-9}}{\log_{10} \frac{D}{R}} \text{ farads}$$

The value of capacitive reactance can be determined from the relationship

$$X_C = \frac{1}{2\pi f C} \text{ ohms}$$

and

$$X_{C_{\text{per mile}}} = \frac{\log_{10} \frac{D}{R}}{2\pi f(38.8 \times 10^{-9})} \text{ ohms per conductor to neutral}$$

where D and R are the distance between conductors and the radius of the conductor, respectively, and are all expressed in inches.

The current per conductor caused by the capacity of the line is

$$I = \frac{V}{X_C}$$

$$I = 2\pi f C V$$

where X_C is capacitive reactance and V is line to neutral voltage. This current (usually called the charging current) increases with the length of the line, since the capacitive reactance decreases with an increase in the length of line.

4. Circuit Calculations. Problems in voltage regulation, efficiency, power factor correction, and voltage control are some of the usual problems encountered in dealing with transmission lines. These problems are attacked in different ways, depending upon the magnitudes of the

values of R, $X_{\mathcal{L}}$, and X_C for the lines. To solve problems considering the constants distributed as in the actual case, hyperbolic functions are necessary. If the constants are considered as lumped, two types of circuits can be used. These two equivalent circuits are the "pi" (π) and "T."

Figure 1–14a shows the π circuit and Fig. 1–14b the T circuit. For the π line, the capacitive reactance of the line is considered as two lumped constants placed at the ends of the line with the total resistance and inductive reactance between them. For the T line, the capacitive reactance is lumped at the center with the resistance and inductive reactance divided on the two sides. The values to be used are the values to neutral, and the constants used for the two conditions give approximately the same final results.

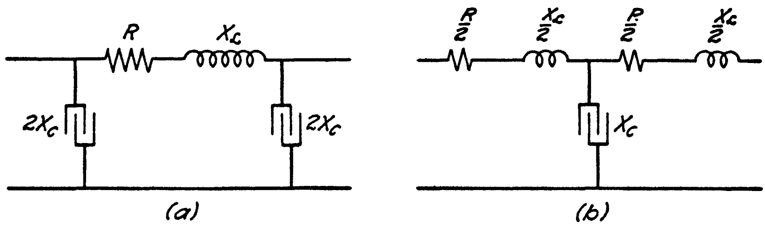


FIG. 1–14. (a) The equivalent π circuit for a transmission line. (b) The equivalent T circuit for a transmission line.

Example a. Using both π and T circuit connections, determine the charging current per phase for a three-phase line 75 miles long if the conductors are 0000 copper with triangular spacing of 72 in. The line-to-line voltage is 66,000 volts. The system frequency is 60 cycles.

Diameter of the conductor is 0.522 in. and radius is 0.261 in.

The inductive reactance per conductor is

$$X_{\mathcal{L}} = 75\left[377\left(0.0805 + 0.74 \log_{10}\frac{72}{0.261}\right)10^{-3}\right] \text{ ohms}$$

$$X_{\mathcal{L}} = 51.4 \text{ ohms}$$

The capacitive reactance to neutral per mile of conductor is

$$X_C = \frac{\log_{10}\dfrac{72}{0.261}}{377(38.83 \times 10^{-9})} \text{ ohms}$$

$$X_C = 165{,}000 \text{ ohms}$$

$$X_C \text{ for 75 miles} = \frac{165{,}000}{75} = 2210 \text{ ohms}$$

The resistance per conductor for 75 miles is

$$R = 75 \times 0.278 = 20.85 \text{ ohms}$$

The voltage to neutral is

$$\frac{66{,}000}{\sqrt{3}} = 38{,}105 \text{ volts}$$

Figure 2–14 shows the diagram of the π circuit and gives the values of R, X_L, and X_C. Figure 3–14 shows the T circuit diagram and the corresponding values for the circuit parameters.

The values of R and X_L are very small when compared to the capacitive reactance and for all practical considerations may be neglected when finding the charging current. Neglecting the values of R and X_L, the charging current in both types is

$$I = \frac{66{,}000}{\sqrt{3} \times 2210} = 17.5 \text{ amp}$$

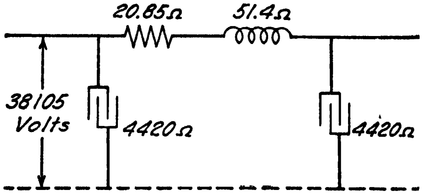


FIG. 2–14. Equivalent π circuit with values for the parameters.

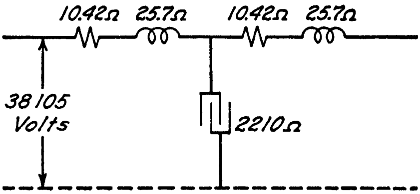


FIG. 3–14. Equivalent T circuit with values for the parameters.

Example b. The electrical load of a small town is 400 kw at 80 per cent power factor lag. If the voltage at the substation is 11,000 volts, what is the voltage at the generating station 15 miles away? What is the power loss in the line and efficiency of transmission? The load is single phase. The constants per mile of transmission line conductor are $R = 0.2$ ohm; $X_L = 1.0$ ohm. Neglect the effect of capacity between lines.

$$\text{Total resistance of line} = 2 \times 15 \times 0.2 = 6 \text{ ohms}$$

$$\text{Total reactance of line} = 2 \times 15 \times 1 = 30 \text{ ohms}$$

$$\text{Current } I = \frac{400{,}000}{11{,}000 \times 0.8} = 45.45 \text{ amp}$$

The voltage at the generator $\bar{V}_g = \bar{V} + \bar{I}\bar{Z}_L$.

If

$$\bar{V} = 11{,}000 + j0 \quad \text{then} \quad \bar{I} = 36.36 - j27.27 \qquad \bar{Z}_L = 6 + j30$$

$$\bar{V}_g = (11{,}000 + j0) + (36.36 - j27.27)(6 + j30)$$

$$\bar{V}_g = (11{,}000 + j0) + (1036.26 + j927.18)$$

$$\bar{V}_g = 12{,}036.26 + j927.18 \qquad V_g = 12{,}041 \text{ volts}$$

$$\text{Power loss in the line} = I^2R = 45.45^2 \times 6 = 12{,}394 \text{ watts}$$

$$\text{Power input at generator} = 400{,}000 + 12{,}394 = 412{,}394 \text{ watts}$$

$$\text{Efficiency of transmission } \frac{400{,}000}{412{,}394} = 97 \text{ per cent}$$

5. System Protection. All electrical systems must be protected against abnormal disturbances which may cause sudden changes in voltage and current. These disturbances must be removed from the system as quickly as possible in order to prevent serious damage to the equipment. These disturbances may be grouped according to cause, as (1) circuit switching, (2) lightning, and (3) insulation failures and short circuits.

The protection against lightning, and the abnormal voltage produced by it, is provided by an arrester, the function of which is to protect the system by providing a low resistance path to ground during the period of disturbance. The ideal lightning arrester would be one which, under normal conditions, would have an infinite resistance to ground but, during an abnormal period, would instantly change to a negligible resistance. The fuse provides protection against switching, overloads, and short circuits. It was the first protective device used and continues to be considered a good protection against overload currents. Circuit breakers and oil switches are operated by solenoids, which are controlled by sensitive relays. The circuit breakers and oil switches with their auxiliary equipment may be designed to operate on either voltage or current variations.

All protective devices must have a degree of sensitivity and accuracy in proportion to the importance of the circuit to be controlled. A protective device must have the ability to differentiate between unusual and normal situations, functioning properly with speed and accuracy for all conditions above and below its operational setting. The transient variation, which exists for a fraction of a second, may be caused by the closing of a switch or the starting of a motor; however, it is necessary for the protective equipment to have a time setting in excess of this transient period in order to prevent an interruption in service. This time lag reduces the number of unnecessary interruptions in service.

6. Fuses. The fuse is the most common type of current protective device used in electrical installations. In both lighting and power installations for factories and buildings, each circuit must be protected by a fuse. The simple fuses used on circuits up to 550 volts are composed of an alloy of low melting point enclosed in a fiber tube which is packed with a non-combustible material. The ends of the fiber tubes terminate in terminals of various designs which complete the circuit between the alloy fuse material and the terminals of the electrical circuit.

Fuses are available for all voltages up to 69 kilovolts. Figure 4–14 shows the liquid fuse used in protecting high voltage systems. In this type of fuse, the link is enclosed in a tube filled with non-inflammable

liquid and, in order to increase the interrupting power of the fuse, the link may be held in tension by a spring. When this type of fuse fails, the spring collapses and opens the gap, aiding the liquid in extinguishing the arc.

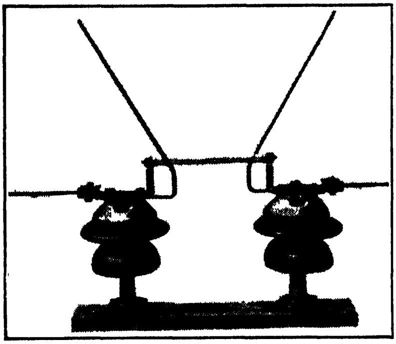


FIG. 4–14. Horn gap fuse for outdoor service. (Courtesy of Railway and Industrial Engineering Co.)

The horn gaps permit the use of the ordinary link fuse where the tension spring is not used. The figure shows a fuse mounting for a 37,000-volt circuit. Fuse disconnect switches are available for systems of 15,000 volts and less, which act in the dual capacity of switch and fuse. The fuse link is protected by a fiber tube which, with the fuse, serves as the blade of the air disconnect switch. Many fuses are so designed that there is some visual indication when blown. Figure 5–14 shows a fuse cutout of this type. When the fuse link in the fiber tube is blown, the automatic latch at the bottom is released, permitting the fuse holder to drop out of the socket. The open position of the fuse holder indicates that the fuse is blown. The switch-hook socket at the top is used in operating the fuse cutout as an air switch. The small gaps on the spring holder at the top of the fiber tube act as horn gaps in aiding to break any arc which may occur as the fuse holder falls out.

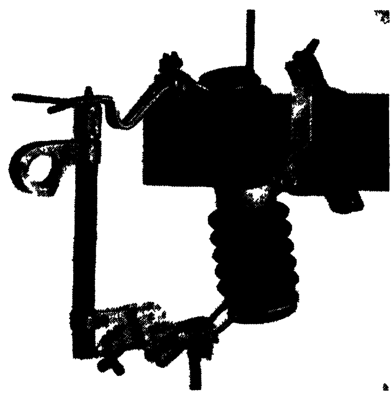
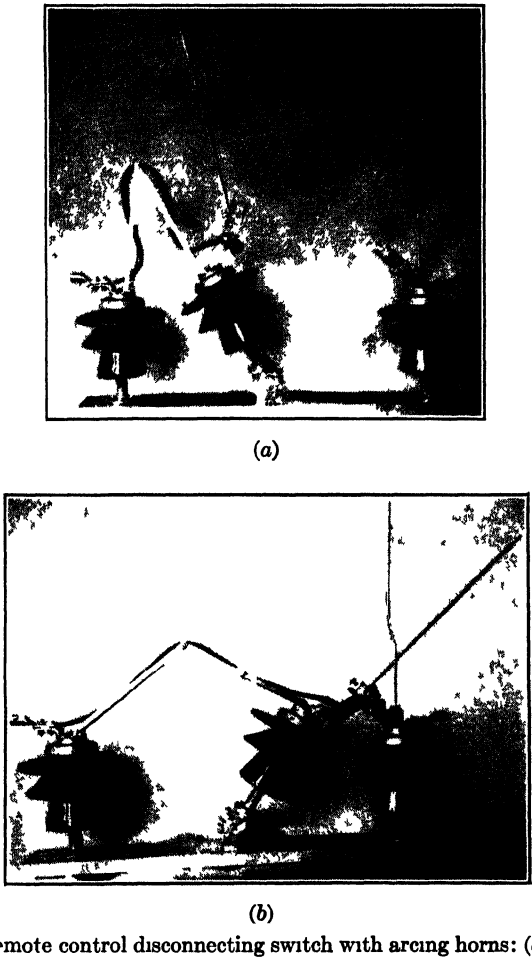


FIG 5–14. Closed position of an open fuse cutout with drop-out fuse holder. Rating 7500/12,500 wye volts, 60 amperes (Courtesy of General Electric Co.)

When a fuse is used on a high voltage system, it is to provide additional protection against short circuits and lightning disturbances. The overload protection is furnished by the relays and circuit breakers.

7. Switches. Switches are used to interrupt the electrical system and may range from the simple types used to control electric lights in the home to elaborate equipment used in high voltage transmission systems. The switches used in disconnecting lights and electrical equipment are

rated from five amperes to thousands of amperes. Switches having a small current capacity usually operate rapidly in order to protect the small contacts. The larger capacity switches are used to disconnect heavy current equipment from the supply lines and, if frequently



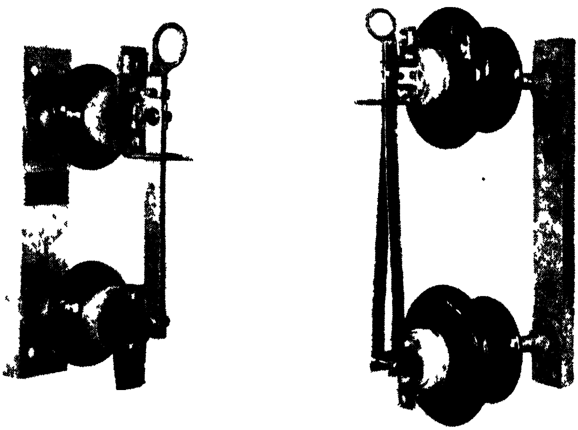
(a)

(b)

Fig. 6–14. Remote control disconnecting switch with arcing horns: (a) switch open; (b) switch closed Rating 46 kv, 600 amp. (Courtesy of Westinghouse Electric Corp)

operated when carrying large currents, the switch contacts are operated in oil to aid in disrupting the arc rapidly. In the intermediate group, such as feeder switches and motor switches, the knife-blade switch is used. It must have a capacity rating sufficient to permit the interrupting of the rated current for the circuit.

Air-break disconnect switches are generally used to open circuits after the system has been disconnected from the source. The circuit is never opened at the air switch when carrying a load, because the current would produce an arc which would burn the contacts. Figure 6–14 shows the outdoor type of air switch with horn-gap protection. The arc horns are in direct contact while the switch is being opened or closed and this prevents burning of the contacts. The horn gap also helps to interrupt the arc after it has been established, in that the further open-



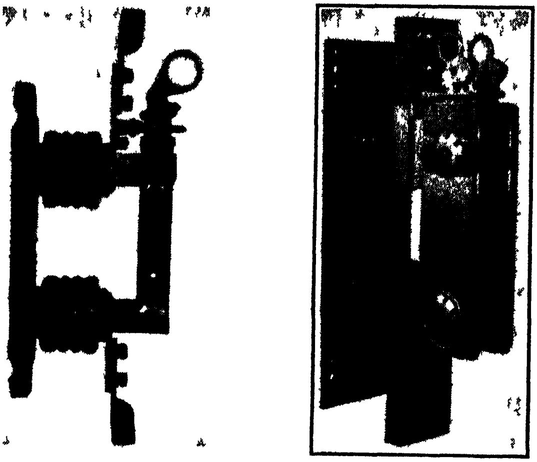
(a) (b)

FIG. 7–14. Outdoor disconnect switch: (a) 7500 volts, 200 amperes; (b) 34,500 volts, 200 amperes. (Courtesy of General Electric Co.)

ing of the switch causes the arc to be drawn out, and the heat produced causes the arc to follow up on the horn gap. This double action on the arc causes it to increase in length to a point where it is broken, and current ceases to flow.

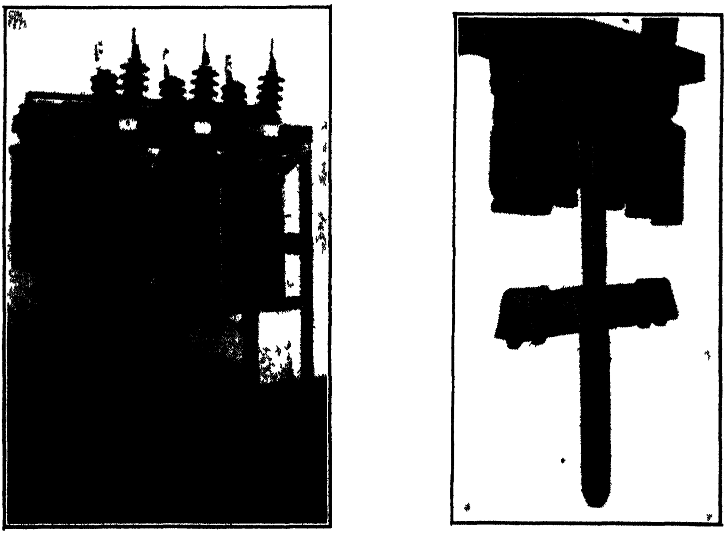
Figure 7–14 shows two outdoor disconnecting switches of different voltage ratings and the same current capacity. Switches of this class are used to isolate sections of the electrical system and are not opened when a current is flowing in them. These switches are used in conjunction with oil-circuit breakers.

For indoor service the air disconnect switch is seldom provided with such heavy insulator supports because it is not exposed to ice, rain, and snow. Figure 8–14 shows two indoor disconnect switches of different voltage and current ratings. Switches of this type are used widely to isolate feeders and transformer banks on the operating bus in substations and small generating stations.



(a) (b)

FIG 8–14 Indoor disconnect switch (a) rating, 15,000 volts, 600 amperes; (b) rating, 7500 volts, 3000 amperes (Courtesy of General Electric Co)



(a) (b)

FIG. 9–14. Mounted oil circuit breaker: (a) 23,000 volts, 600-ampere oil circuit breaker; (b) oil circuit breaker contacts. (Courtesy of General Electric Co)

Fig. 10–14 Oil blast breaker rated at 200 kilovolts Each tank contains one single-pole breaker. (Courtesy of General Electric Co)

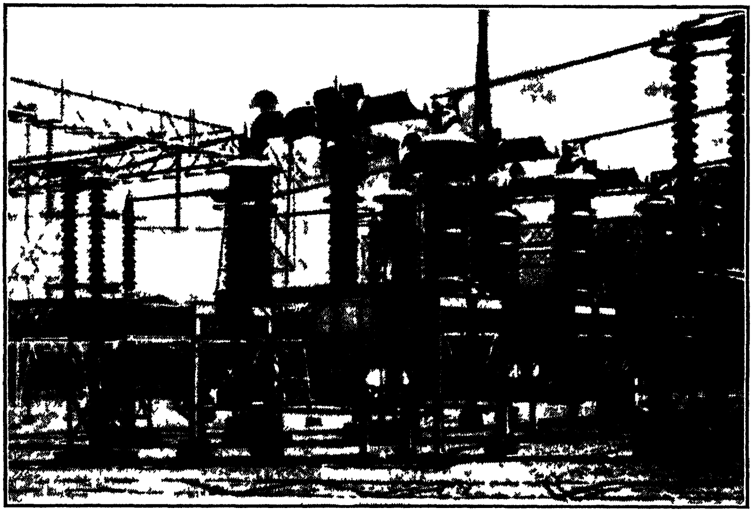
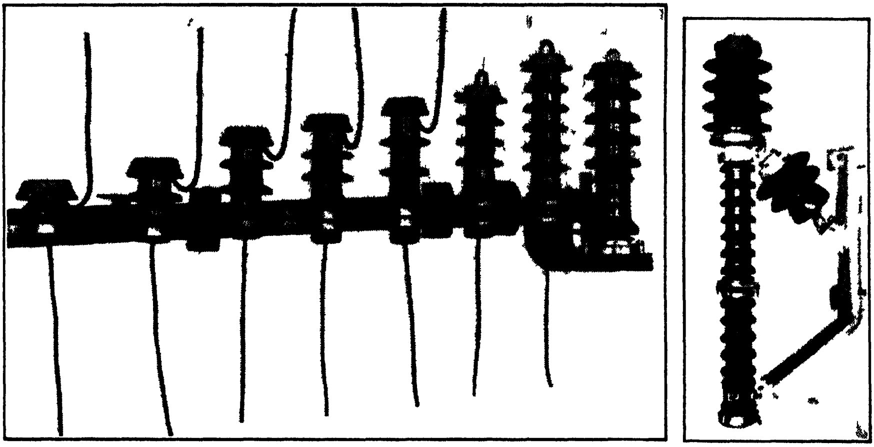


Fig. 11–14. Impulse oil circuit breaker Rating: 287,500 volts, 1200 amperes (Courtesy of General Electric Co.)

8. Circuit Breakers. The circuit breaker is used to interrupt the circuit while the current is flowing. Circuit breakers are manufactured in such low capacities that ordinary 15-ampere residence lighting circuits may be equipped with them.

For low voltages, the circuit breaker contacts are opened in the air but for high voltages the contacts are submerged in oil to prevent arcing across the gap between the contacts. Compact circuit breakers for interrupting high voltage systems of large capacities are constructed,



(a) (b)

FIG. 12–14. (a) Lightning arresters for 3, 6, 9, 12, 15, 20, 25, and 30 kilovolts; (b) 50,000-volt arrester with mounting bracket. (Courtesy of Westinghouse Electric Corp.)

using oil around the contacts. Each pole of the breaker shown in Fig. 9–14 is enclosed in a steel tank containing an insulating oil. When the circuit breaker is closed, the arm is engaged with the contacts and held in position but, when the breaker is released, the energy stored in springs and the weight of the moving contact cause the contacts to open rapidly. Figure 9–14*b* shows the breaker contacts in the open position.

To increase the interrupting rating of a circuit breaker, two new principles have been followed in the past few years. These are known as the *impulse oil blast* and *de-ion grid* principles. The first consists in supplying a blast of oil at high pressure into the arc stream, thereby increasing the circuit breaker interrupting capacity. The de-ion grid consists of alternate layers of insulating and magnetic materials arranged in such a way that the arc is drawn into a narrow slot and held by the magnetic field. As the arc ionizes the path of the oil, it (the arc)

goes out as the alternating current passes through zero in its cycle, and cannot be reestablished by the voltage of the system. Figure 10–14 shows a group of oil circuit breakers on a 220,000-volt system. Each tank contains one single-pole breaker. Figure 11–14 shows the impulse oil circuit breaker installed in the outdoor substation on the Boulder Dam to Los Angeles transmission line.

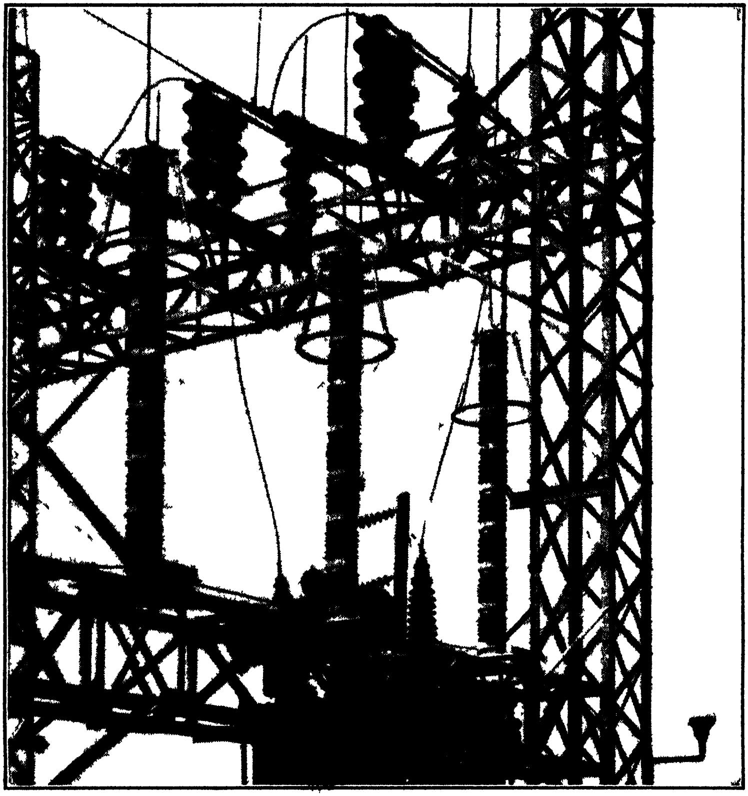
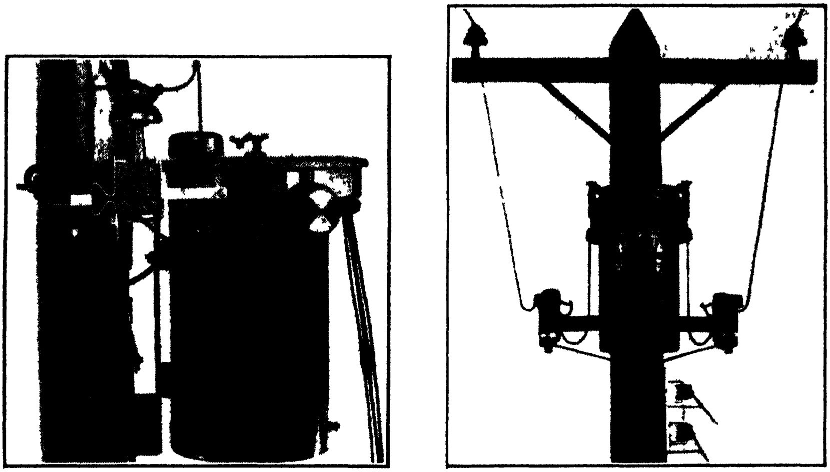


FIG. 13–14. Three-phase arrester protecting 115-kilovolt substation equipment. (Courtesy of General Electric Co.)

9. Lightning Arresters. The purpose of lightning arresters is to protect the insulation of the system by providing a low resistance path to ground when excessive voltages are produced in the lines by a lightning stroke. The arrester is connected between line and ground and under normal voltage, the resistance to ground is very high and no appreciable current flows. Essentially the arrester consists of an air gap and a high resistance in series. When the voltage exceeds the rating of the gap, the

gap breaks down and a current flows to ground. As the voltage decreases, the arc is interrupted and the device is restored to its original condition. The time of operation of an arrester is from 10 to 50 microseconds. As the voltage of the system is increased, the size of the lightning arrester is increased in order that the correct series gap and resistance can be obtained. Figure 12–14 shows a group of lightning arresters for voltages from 3 to 50 kilovolts.

Figure 13–14 shows the installation of arresters on a 115-kilovolt three-phase system. The shielding ring at the top of each arrester is to protect it in the event of a flashover to ground. This type of arrester is



(a) (b)

FIG. 14–14. (*a*) Combination cross-arm mounting of arrester and fuse cutout. (*b*) Typical installation showing fuse cutouts, lightning arresters, and secondary fuse cutouts. (Courtesy of General Electric Co.)

built in sections, with each section rated at approximately 20,000 volts. Figure 14–14*a* shows the mounting of a distribution transformer, lightning arrester, and fuse cutout on a cross arm, the arrester being connected on the line side of the cutout. Figure 14–14*b* shows a similar system with the secondary side protected by fuse cutouts which are mounted on the secondary terminals of the transformer. In line construction, high voltage is placed on the pole above low voltage for safety.

Figure 15–14 shows the most recent development of a distribution transformer with arresters and grounding gap.

10. Insulators. Porcelain, Pyrex glass, and glass are the insulator materials used in electrical transmission and distribution. Porcelain is the best of the insulators and is used widely for high voltage systems.

However, Pyrex glass and glass are used along with porcelain for lower voltages. The surface of porcelain insulators is glazed and shaped to give high flashover characteristics. All good insulators will arc over before puncture and mechanical injury occurs, and a wet arc-over test is usually made to determine the voltage rating of the insulator. This test determines the flashover voltage of the insulator, when it is subjected to water falling at the rate of 1 inch every 5 minutes.

11. Distribution. Power is supplied to most of the lighting and small power loads over three-wire distribution circuits. The Edison three-

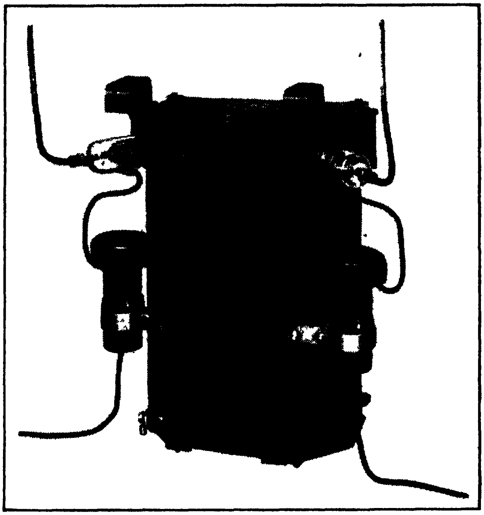


FIG. 15–14. Distribution transformer with lightning arresters and grounding gap. (Courtesy of Allis-Chalmers Mfg. Co.)

wire system for direct current and the three-wire single-phase system for alternating current supply the 110 to 220-volt loads. Both systems can supply 110 volts from either of the two outside lines to neutral and 220 volts between outside lines. The neutral line is usually grounded and should not be fused. The fuse protection in the outside lines is sufficient for both the 110-volt and 220-volt loads and the danger of the neutral fuse burning out and giving unbalanced voltages is eliminated.

The neutral wire is usually the same size as the other line wires but, if the individual loads are balanced on the two 110-volt sources, the neutral current will be zero. The neutral current under normal loading is not equal to the line current as the load is usually divided between the two lines and neutral. The neutral, therefore, carries the vector difference of the two line currents.

Table II–14 shows the comparative copper requirements for the various distribution systems, the two-wire circuit being considered as the basis.

TABLE II–14

COPPER REQUIREMENTS FOR SYSTEMS OF DISTRIBUTION

System	Voltages	Copper for Constant Voltage Drop [1]	Copper for Constant Load [2]
Two-wire a-c and d-c	100%	100%	100%
Three-wire a-c and d-c	100–200%	37.5%	75%
Three-phase 3-wire	100%	75%	87%
Three-phase 4-wire	100–173%	33.3%	67%
Two-phase 4-wire	100%	100%	100%
Two-phase 3-wire (Neutral 141% of outside wire)	100–141%	73%	85%

[1] Constant voltage drop, load and voltage—adequate and safe.
[2] Constant load and voltage—only short runs—safe.

In residential districts of most communities, the power output of the distribution transformer is transmitted over the three-wire secondary to the consumer.

The consumers loads are connected to the secondary system so that the total load on the transformer at any time is nearly balanced. This is accomplished by a complete study of the total load, its diversification and period of use.

Example c. In the circuit diagram (Fig. 16–14*a*), the individual loads are (1) 100-watt lamp; (2) 100-watt lamp; (3) 500-watt toaster; (4) 25-watt lamp. Find the current in each lead; and if the neutral is broken at "x" what is the voltage across the 25-watt lamp?

A 100-watt lamp requires 1 amp at 100 volts, and its resistance is 100 ohms.
A 500-watt toaster requires 5 amp, and its resistance is 20 ohms.
A 25-watt lamp requires 0.25 amp, and its resistance is 400 ohms.
The circuit resistances are as shown in Fig. 16–14*b*.

$$I_{AA'} = 1 + 5 + 1 = 7 \text{ amp}$$
$$I_{DE} = 0.25 \text{ amp}$$
$$I_{CF} = 7 - 0.25 = 6.75 \text{ amp}$$

When the neutral is opened, the circuit is shown by Fig. 16–14c.

$$\text{The current } I_{AB} = I_{BC} = I_{CD} = I_{DE} = \frac{200}{414.3} = 0.482 \text{ amp}$$

$$V_{CD} = 400 \times 0.482 = 192.8 \text{ volts}$$

and the lamp would burn out.

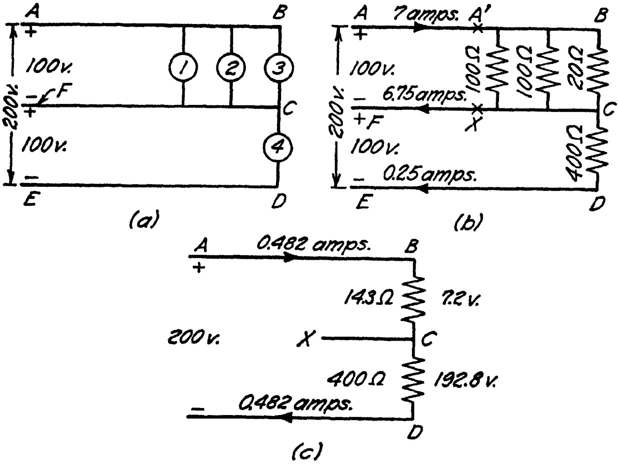


FIG. 16–14. (a) Three-wire circuit. (b) Equivalent circuit for the three-wire circuit. (c) Three-wire system with neutral open.

Example d. Two loads, having equivalent impedances of $2 + j5$ and $4 + j8$, respectively, are connected to a three-wire 110- to 220-volt single-phase supply by wires having an impedance of $1 + j0$ each. Find the voltage, current, and power in each load. Draw a vector diagram if the voltages for the circuit diagram (Fig. 17–14a) have the complex expressions

$$\bar{V}_{AB} = 110 + j0 \qquad \bar{V}_{BC} = 110 + j0 \qquad \bar{V}_{AC} = 220 + j0$$

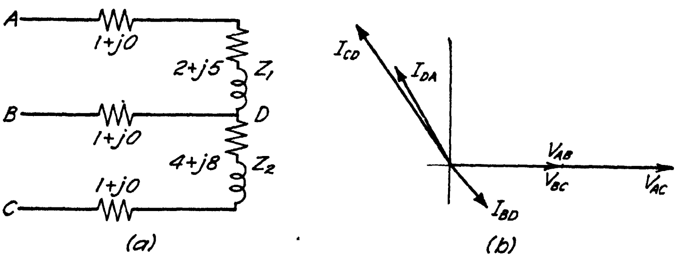


FIG. 17–14. (a) Three-wire circuit. (b) Vector diagram for circuit in (a).

By using Kirchhoff's Laws, the following equations can be written

Loop $ABDA$: $(110 + j0) + \bar{I}_{BD}(1 + j0) + \bar{I}_{DA}(5 + j8) = 0$

Loop $BCDB$: $(110 + j0) + \bar{I}_{CD}(3 + j5) + \bar{I}_{DB}(1 + j0) = 0$

Point D: $\bar{I}_{CD} + \bar{I}_{BD} + \bar{I}_{AD} = 0$

From the current equation,

$$\bar{I}_{BD} = \bar{I}_{DA} - \bar{I}_{CD}$$

$$\bar{I}_{DB} = \bar{I}_{CD} - \bar{I}_{DA}$$

Substituting in the voltage equations and simplifying,

Loop $ABDA$: $110 + j0 + \bar{I}_{DA}(6 + j8) - \bar{I}_{CD}(1 + j0) = 0$

Loop $BCDB$: $110 + j0 - \bar{I}_{DA}(1 + j0) + \bar{I}_{CD}(4 + j5) = 0$

$\bar{I}_{DA}$ is eliminated and the final expression is

$$\bar{I}_{CD} = \frac{-770 - j880}{-17 + j62} = -10.03 + j15.18$$

Substituting and solving for $\bar{I}_{DA}$ and $\bar{I}_{BD}$,

$$\bar{I}_{DA} = -5.99 + j10.51$$

and

$$\bar{I}_{BD} = 4.04 - j4.65$$

$I_{CD} = 18.18$ amp $V_{Z_1} = I_{CD}Z_1 = 18.18 \times 5.38 = 97.8$ volts

$I_{DA} = 12.10$ amp $V_{Z_2} = I_{DA}Z_2 = 12.10 \times 8.95 = 108.3$ volts

$I_{BD} = 6.17$ amp

$$P_{Z_1} = (I_{CD})^2 \times 2 = (18.18)^2 \times 2 = 661 \text{ watts}$$

$$P_{Z_2} = (I_{DA})^2 \times 4 = (12.1)^2 \times 4 = 586 \text{ watts}$$

BUILDING POWER DISTRIBUTION

The design of both primary and secondary power distribution is the task of the specialist in electrical engineering, whereas the design of the wiring for circuits in a building may become the duty of some person or persons designated as the engineer of the plant, the administrative head of the installation, or maintenance group. The design of these elementary circuits is governed by the fundamental principles of circuits.

Since these circuits are hazardous both to the building and to the lives of the occupants, there are two prime factors to be considered, namely, safety and performance. The first is governed by the rules of the National Board of Fire Underwriters * through the "National Electrical Code" and the city and state laws. The latter performance is strictly an electrical problem governed by the necessity of delivering sufficient voltage to insure proper functioning of the equipment.

The National Electrical Code originates with, and is revised by, the Electrical Committee of the National Fire Protection Association. The main committee is composed of subcommittees and has specialists in

* Tables of motor currents and wire sizes with allowable current-carrying capacity are taken from the 1940 code.

the fields of inspection and practice associated with it. The rulings of this committee are approved by the American Standards Association and the National Board of Fire Underwriters, and this latter organization prints and distributes the various editions of the code to the fire marshals of the states and the inspectors of the municipalities. Besides these basic rules and regulations, the states and cities have laws dealing with special problems deemed necessary for regulation within limited jurisdictions. A lighting or power circuit installed in a building according to these rules and properly maintained will never be a hazard. The hazard lies in the equipment or in faulty circuit installation or maintenance.

It is essential that all installations be made according to the latest code, which is issued in revised form approximately every two years.

If electrical equipment is to function properly, the voltage must be maintained within 10 per cent of the rated voltage of the load and, in electrical lighting, the voltage should be within 5 per cent of the rated voltage (much better results if it is within 3 per cent). In motors, the line current is about 11 per cent of normal current within plus or minus 10 per cent of the rated voltage, whereas, in the electric lamp, the lumen output of the lamp is reduced 10 per cent with 3 per cent voltage drop, and 15 per cent with 5 per cent voltage drop. In all calculations for voltage regulation of a wiring system, the variation in voltage at the supply center must be considered.

12. Design Methods and Accuracy. It is possible to design building distribution systems in the same manner as shown in problems on electrical circuits. Figure 18–14 shows the basic circuit and the vector diagram for such a circuit. It is necessary only to determine the voltage V_1 at the load and the voltage drop in per cent will be

$$\frac{V - V_1}{V_1} \times 100$$

where V_1 is the rated voltage across the load and V is the voltage at the source. A general solution may be obtained by combining the individual line resistance and reactance as shown in Fig. 18–14*b* and, from the vector diagram in Fig. 18–14*c*, the load voltage will be

$$\bar{V}_1 = \bar{V} - \bar{I}\bar{Z}_L$$

An accurate solution of this problem is

$$\bar{V} = \bar{V}_1 + \bar{I}\bar{Z}_L$$

$$\bar{V} = \bar{V}_1 + \bar{I}(R_L + jX_L)$$

which can be solved numerically when the rating of the machine is known and the effective resistance and reactance of the wires for the particular size and spacing have been determined. In the normal light and power circuit solution, this degree of accuracy is not necessary, for the available wire sizes differ by a large percentage and it is necessary to choose the larger wire to be sure of a sufficient safety factor. It is a simple matter to check the voltage drop on a wire which has been installed but, where it is necessary to choose a wire for a definite voltage drop, the problem is one of "trial and error." To reduce the labor in-

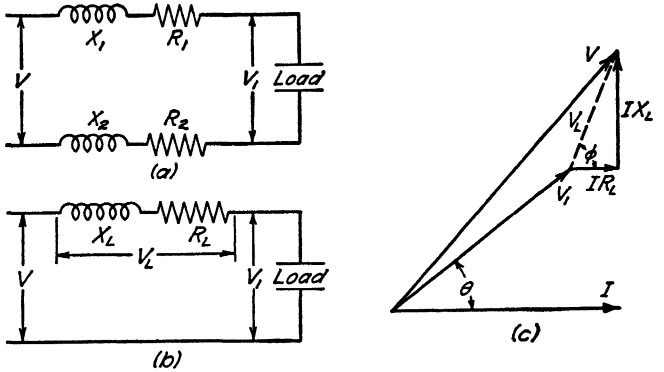


FIG. 18–14. A typical distribution circuit from source to sink.

volved, many rule-of-thumb methods have been devised for selecting proper wire size and, although each method neglects some factors necessary for accurate results, all are close enough for practical purposes.

13. Composite Diagrams. By the reduction of the voltage drops of resistance and reactance to percentage, it is possible to construct diagrams which consider both the power factor of the line and the load. Figure 19–14 shows the graphical method of constructing such a diagram, which may be modified to solve single-phase, two-phase, or three-phase problems. The per cent resistance and reactance are

$$\textit{Per cent resistance drop} = \frac{\text{rated current} \times \text{resistance}}{\text{rated load voltage}} \times 100$$

$$\textit{Per cent reactance drop} = \frac{\text{rated current} \times \text{reactance}}{\text{rated load voltage}} \times 100$$

These definitions are important in circuit and equipment calculations.

To use the diagram (Fig. 19–14), it is necessary only to lay off the per cent power factor, which is $\cos\theta \times 100$, and erect a perpendicular to the inside or 100 per cent circle. Then, construct the per cent re-

sistance drop on the horizontal and the per cent reactance drop on the vertical to form a continuous line, starting with per cent power factor.

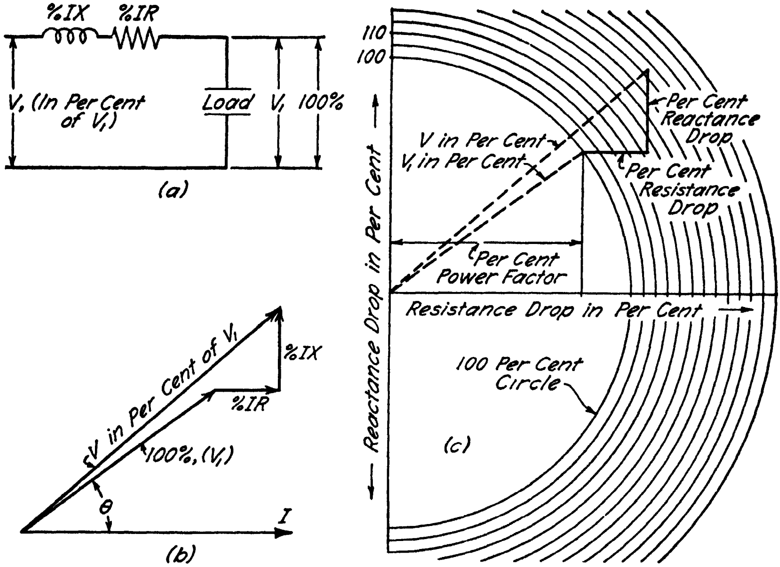


FIG. 19–14. A Mershon diagram for determining the per cent of voltage drop in a circuit, with load voltage as a base.

The point where the reactance drop line intercepts one of the concentric per cent voltage drop circles gives the per cent drop of the circuit. Diagrams of this type are called Mershon diagrams.

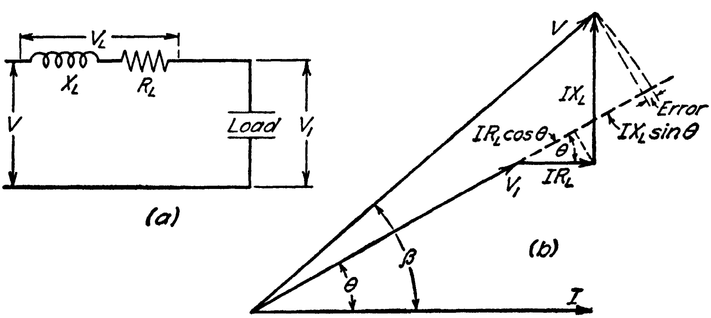


FIG. 20–14. Determination of voltage drop by projecting the resistance and reactance drops upon the load voltage extended.

14. Formula Method. By modifying the diagram in Fig. 18–14c, it is possible to develop a relatively simple formula for direct substitution. As shown in Fig. 20–14, there is an error in this method but, as

pointed out in a previous paragraph, this may be neglected because of the latitude in wire sizes. The error introduced is caused by neglect of the vertical component in the triangle of which V is the hypotenuse. If this component is neglected, V and V_1 lie on the same straight line and the angles θ and β are considered equal in value. This is not serious, for under usual conditions θ and β are approximately equal. Projecting on the rated voltage the components of IR and IX of the equipment gives

$$V_L = I(R_L \cos\theta + X_L \sin\theta)$$

as shown in Fig. 20–14, where V_L is the voltage drop of the system; R_L is the resistance of the line; X_L is the reactance of the line; and the

TABLE III–14

TABLE OF RESISTANCE AND INDUCTIVE REACTANCE[1] OF COPPER WIRE
(Per 1000 ft at 77° F)

Gage Number	Resistance at 60 Cycles	Reactance at 60 Cycles Distance between Centers of Conductors (Inches)									
		1	2	4	6	8	10	12	24	36	48
14	2.666	0.085	0.101	0.117	0.126	0.133	0.138	0.142	0.157	0.167	0.174
12	1.680	0.079	0.095	0.112	0.121	0.128	0.133	0.137	0.153	0.162	0.168
10	1.056	0.074	0.090	0.106	0.115	0.122	0.127	0.131	0.147	0.157	0.163
8[2]	0.6760	0.069	0.085	0.101	0.110	0.117	0.122	0.126	0.142	0.151	0.158
6	0.4255	0.064	0.079	0.096	0.105	0.111	0.117	0.121	0.137	0.146	0.152
5	0.3380	0.061	0.076	0.093	0.102	0.109	0.114	0.118	0.134	0.144	0.150
4	0.2676	0.059	0.074	0.090	0.099	0.106	0.115	0.115	0.131	0.141	0.147
3	0.2123	0.053	0.069	0.084	0.097	0.099	0.105	0.109	0.129	0.138	0.145
2	0.1770	0.049	0.066	0.081	0.094	0.097	0.103	0.107	0.126	0.135	0.142
1	0.1328	0.047	0.063	0.079	0.091	0.095	0.100	0.104	0.123	0.133	0.139
0	0.1056	0.046	0.060	0.076	0.089	0.092	0.097	0.101	0.121	0.130	0.136
00	0.0839	0.042	0.058	0.074	0.086	0.089	0.095	0.099	0.118	0.127	0.134
000	0.0666	0.039	0.055	0.071	0.083	0.087	0.089	0.091	0.115	0.125	0.131
200,000	0.0562	0.038	0.053	0.069	0.081	0.085	0.088	0.092	0.114	0.123	0.130
0000	0.0528	0.037	0.052	0.068	0.080	0.084	0.088	0.093	0.113	0.122	0.129
250,000	0.0447	0.034	0.050	0.066	0.079	0.082	0.087	0.091	0.111	0.120	0.127
300,000	0.0378	0.032	0.048	0.064	0.077	0;080	0.085	0.089	0.109	0.118	0.125
350,000	0.0323	0.030	0.047	0.063	0.075	0.079	0.084	0.088	0.107	0.116	0.123
400,000	0.0283	0.029	0.045	0.061	0.073	0.077	0.082	0.086	0.105	0.114	0.121
500,000	0.0227	0.027	0.042	0.059	0.071	0.074	0.080	0.084	0.103	0.112	0.119
600,000	0.0194	0.024	0.040	0.056	0.069	0.072	0.077	0.081	0.102	0.110	0.118
700,000	0.0166	0.023	0.039	0.055	0.068	0.071	0.075	0.079	0.100	0.108	0.116
750,000	0.0155	0.028	0.038	0.054	0.066	0.070	0.075	0.079	0.098	0.107	0.114
800,000	0.0151		0.037	0.053	0.065	0.069	0.074	0.078	0.097	0.106	0.113
900,000	0.0141		0.036	0.051	0.064	0.068	0.072	0.076	0.096	0.105	0.112
1,000,000	0.0120		0.034	0.050	0.063	0.066	0.071	0.075	0.095	0.104	0.111

[1] Tables for 60-cycle, for 25-cycle reactance multiply by 0.417.
[2] Standard Concentric Cable for No. 8 and larger.

angle θ is the power-factor angle of the load. To use this formula it is necessary to have a table of effective resistances and reactances for the various spacings to be considered, and a table of functions for the normal power-factor angles. The resistances and reactances are usually given for 1000 feet of wire. Table III–3 gives the resistance of copper wire per 1000 feet and Table III–14 gives the resistances and reactances, for the usual spacings, for copper wire.

15. Correction Coefficients. It is possible to express the voltage drop in any system, with a fair degree of accuracy, by means of a general formula based on the fundamentals of the electrical circuit, and then, by means of mechanical aids such as diagrams and charts, to reduce the

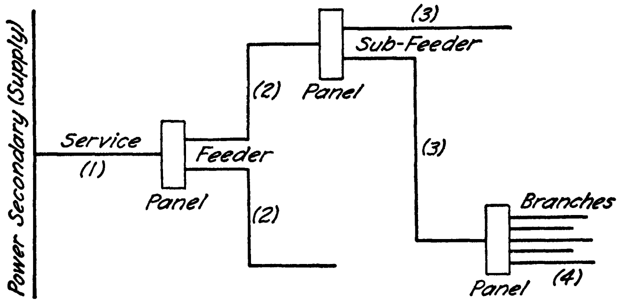


FIG. 21–14. Radial distribution in building wiring.

labor of circuit design to a minimum. The voltage drop in the electrical circuit will be IZ, which may be resolved into two components: that part caused by the resistance, and that part caused by reactance. In the d-c system, the impedance of the system is caused by the resistance alone. By the determination of the proper coefficient, it is possible to calculate the system as if it were a d-c system and then multiply the result by a correction coefficient which provides for the line reactance and load power factor.

Referring to Fig. 18–14c, the resultant voltage in per cent will be equal to

$$
\begin{aligned}
V &= (V_1 \cos\theta + IR_L) + j(V_1 \sin\theta + IX_L) \\
&= V_1 (\cos\theta + j\sin\theta) + I(R_L + jX_L) \\
&= V_1 (\cos\theta + j\sin\theta) + IR_L\left(1 + j\frac{X_L}{R_L}\right) \\
&= V_1 (\cos\theta + j\sin\theta) + IR_L(1 + j\tan\phi)
\end{aligned}
$$

This shows that the resultant voltage will be influenced by the ratio of the reactance to the resistance ($\tan\phi$) and the angle θ, which is the

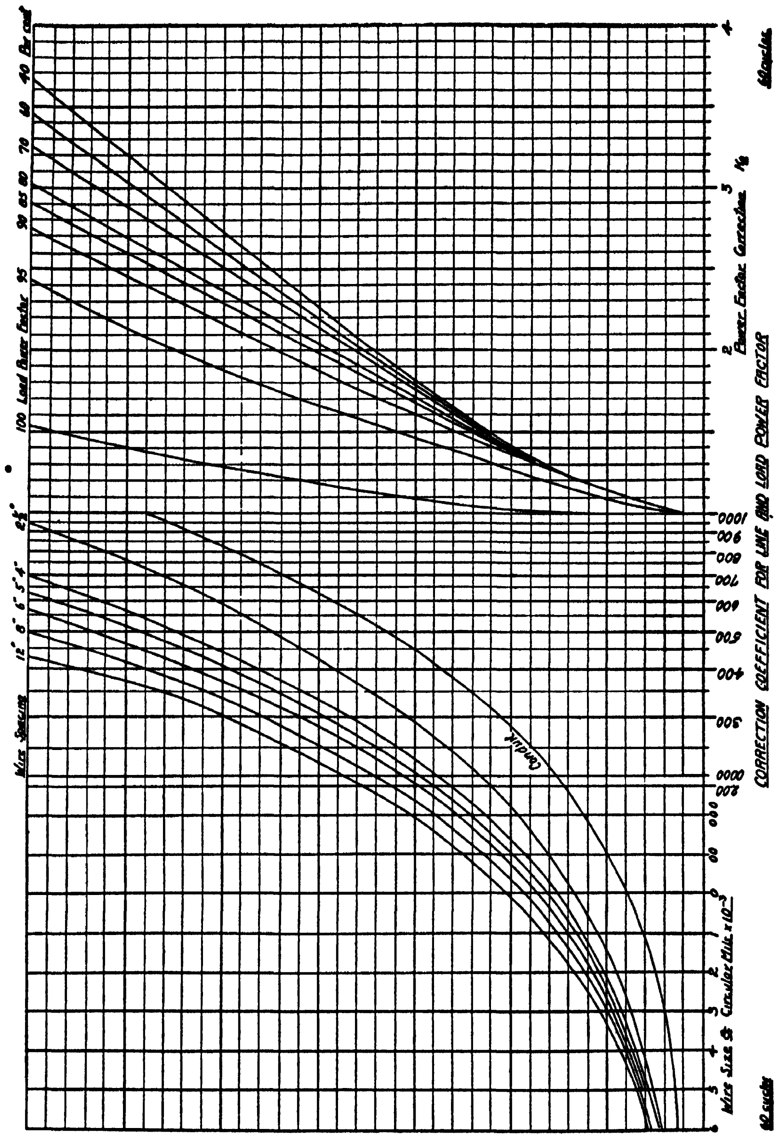


FIG. 22-14. Sixty-cycle correction coefficient (K_2) diagram for line and load power factor.

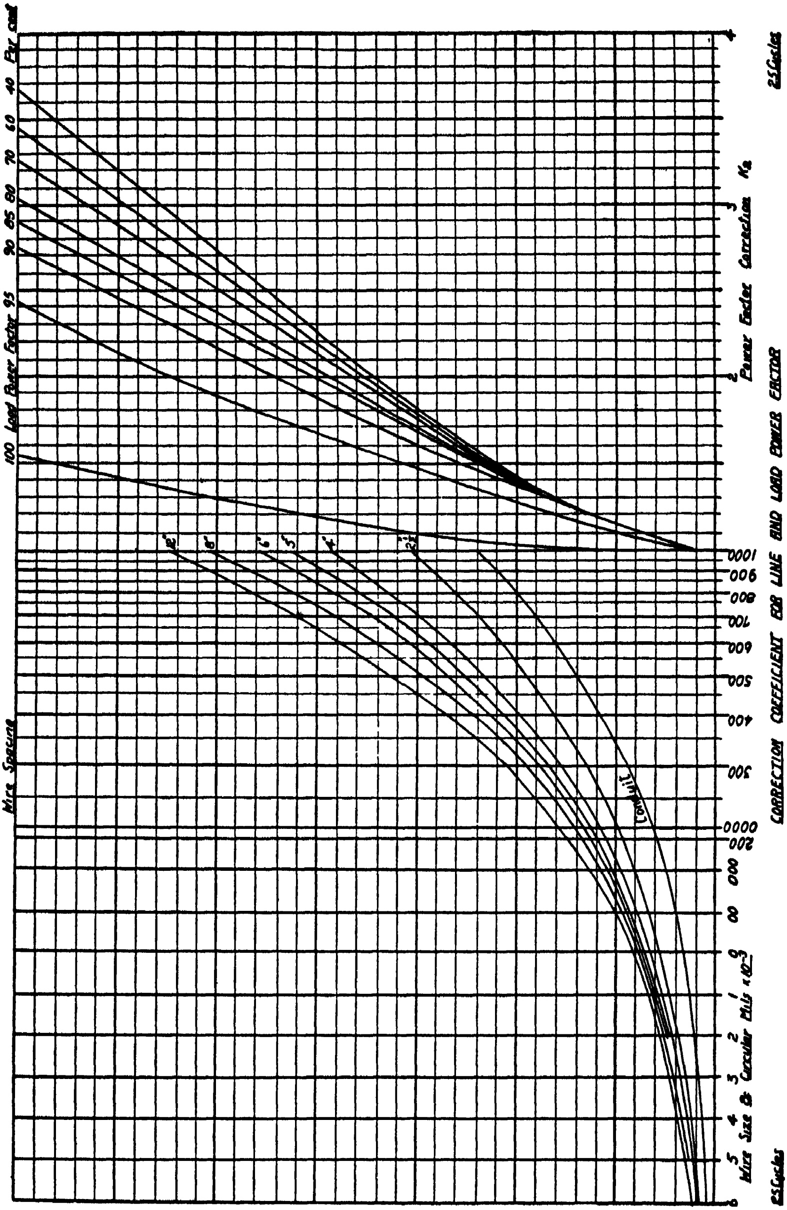


FIG. 23–14. Twenty-five-cycle correction coefficient (K_2) diagram for line and load power factor.

power-factor angle of the load. From tables or individual calculations the ratio X/R may be determined for wires of various sizes and spacing, and, with this as an ordinate and the correction factors as abscissa, a family of power-factor curves may be drawn which will give the correction coefficient K_2, which is used in the method described in the next article for any spacing and wire size with any load power factor. Charts for the determination of this correction are given in Fig. 22–14 and 23–14 (see also Fig. 24–14) for 60 and 25 cycles, respectively. Since

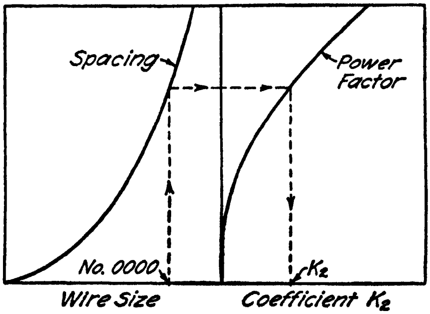


FIG. 24–14. Method of using charts in Fig. 22–14 and Fig. 23–14.

the reactance (which in building wiring is inductive) is directly proportional to the frequency, the correction will be less for 25 cycles than it will be for 60 cycles, other factors remaining constant.

16. Voltage Drop by Formula. The circuits in a building will fall under the following general classifications (Fig. 21–14): (1) service, (2) feeders, (3) subfeeders, and (4) branches. The design of building distribution is in the reverse order. The branches determine the load on the subfeeder; the subfeeders determine the load on the feeders; and the total load of the feeders determines the service needed. The design of the distribution for large factories and particularly for tall office buildings is a problem for the specialist, because the vertical distribution required is similar to a secondary distribution system in a small city.

There are two types of installation, open and concealed wiring, the former being frequently used in factory and mill construction, the latter generally used in commercial and dwelling buildings. There are no definite limits for the use of the various wiring methods for all types are often found in the same structure. Conduit installations, though more sturdy, are also more expensive, and economy of installation is frequently the controlling factor in the choice of the type of installation.

The following systems of wiring for circuits are encountered in building distribution.

Two-wire direct current	For lighting branches and motor circuits
Three-wire direct current	For lighting feeders and for combination lighting and power feeders
Two-wire single-phase alternating current	For lighting branches and small motor circuits
Three-wire single-phase alternating current	For lighting feeders and for combination light and small motor service
Three-wire three-phase	For motors (Lighting may be taken from two of the three wires but this is not the best practice.)
Four-wirc three-phase	Combination lighting and power (The fourth wire is the neutral. Motors, 208 volts; lighting at 120 volts.)

The two-phase system may be either three- or four-wire, but for balanced voltages the four-wire installation is used.

GENERAL VOLTAGE DROP EXPRESSION

The general expression for voltage drop must have factors for modifying the drop caused by resistance, not only for the reactance introduced by the alternating current but also for coefficients for determining the amount of wire that must be considered and the variation in spacing that will be introduced when the three-phase systems are not symmetrical. On a normal spacing of 6 inches, the two lines composing a phase will be 6 or 12 inches apart, depending upon which phase is being considered.

The following is the general expression for the voltage drop in a system:

$$VD = \frac{11 \times I \times D}{\text{circular mils}} \times K_1 \times K_2 \times K_3$$

where 11 is the specific resistance of a circular-mil-foot of copper wire at temperatures encountered in building installations; I is the current in amperes; D is the distance in feet, between the point of supply and the load; K_1 is the coefficient which corrects the distance to the number of feet of wire; K_2 is the coefficient, previously discussed, which corrects for line and load power factor; K_3 is a coefficient depending on the phases in

the system. Table IV–14 gives the values of the various coefficients to be used in solving for voltage drop. In the three-wire d-c and the three-wire, single-phase, a-c system, the voltage drop is given to neutral; in the other systems the voltage drop is the line drop.

TABLE IV–14

CORRECTION COEFFICIENTS TO BE USED WITH FORMULA METHOD

System	K_1	K_2	K_3
2-wire d-c	2	1	1
3-wire d-c (drop to neutral)	1	1	1
2-wire 1ϕ a-c	2	Figs. 22–14 and 23–14	1
3-wire 1ϕ a-c (drop to neutral)	1	Figs. 22–14 and 23–14	1
3-wire 3ϕ a-c (watch spacing)	1	Figs. 22–14 and 23–14	1.73
4-wire 3ϕ a-c (balanced; watch spacing)	1	Figs. 22–14 and 23–14	1.73
4-wire 2ϕ a-c	2	Figs. 22–14 and 23–14	1
3-wire 2ϕ a-c (approximate). Will do for normal problem	Outside wire regular way		
	1	Figs. 22–14 and 23–14	1
	To this add 0.707 drop in common wire		
	1	Figs. 22–14 and 23–14	0.707

Effective spacing:

1ϕ use spacing given.
3ϕ 3-wire spacing 50 per cent greater than given.
3ϕ 3-wire (transposed) spacing 25 per cent greater than given.
3ϕ 4-wire spacing same as 3ϕ, 3-wire.
2ϕ 4-wire spacing as given.
2ϕ 3-wire spacing as given.

The expression considers the ohmic resistance only in a determination of the wire resistance, and this could be corrected for alternating current by an additional coefficient representing the ratio of the effective resistance to the ohmic resistance. Since in practical installations

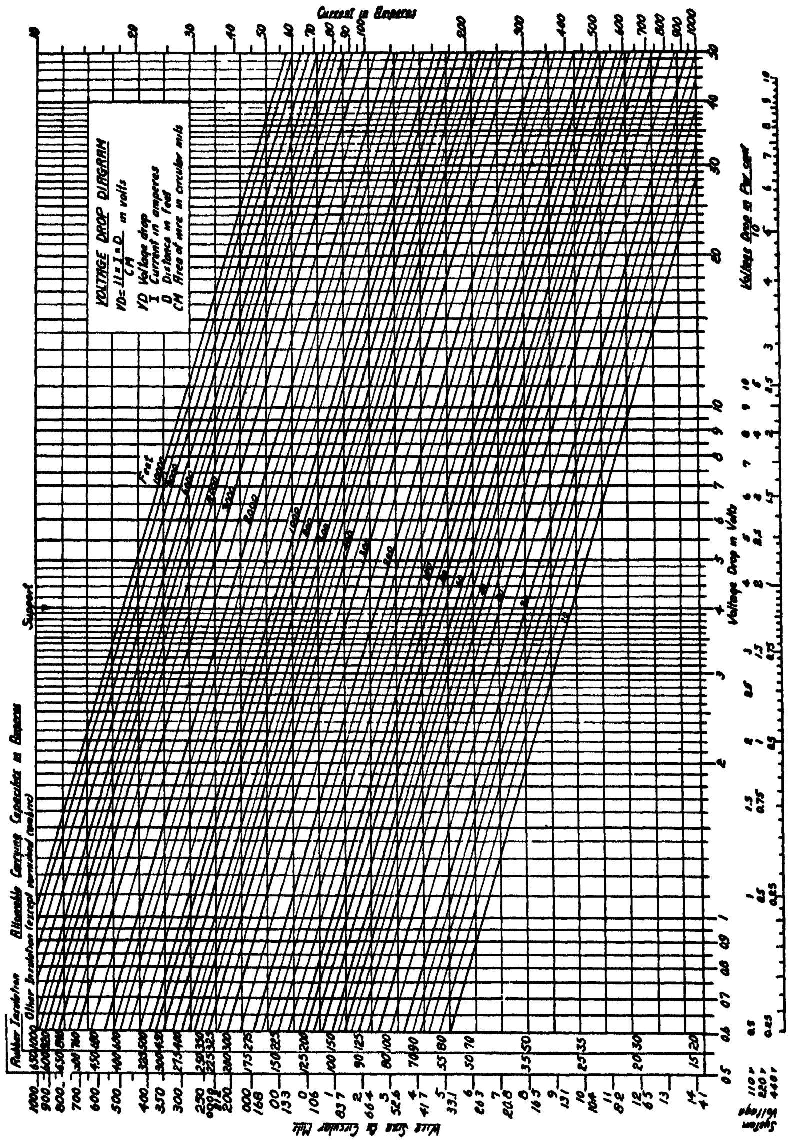


FIG. 25–14. Voltage drop in one line of a d-c system.

this is of minor importance, it has been neglected. Figure 25–14 (see also Fig. 26–14) is a combination graph for determining the value of

$$\frac{11 \times I \times D}{\text{circular mils}}$$

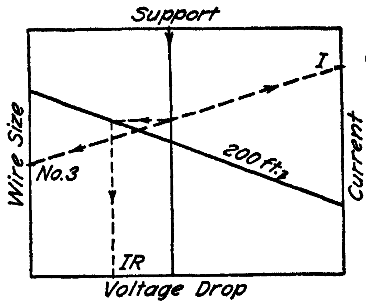


FIG. 26–14. Method of using chart in Fig. 25–14. A straight-edge is placed across the chart connecting the wire size with the current in amperes.

17. Branch Circuits. The following branch circuits may be found in the distribution system of a building, and these are the basic circuits upon which all the design is calculated. These branch circuits may be classified as

a. 15-*Ampere Branch Circuit.* Wire not smaller than No. 14; protection, 15 amperes; maximum load, 15 amperes; portable appliances not in excess of 12 amperes; fixed appliances not in excess of 6 amperes.

b. 20-*Ampere Branch Circuit.* Wire not smaller than No. 12; protection, 20 amperes; maximum load, 20 amperes; appliances not in excess of 15 amperes.

c. 25-*Ampere Branch Circuit.* Wire not smaller than No. 10; protection, 25 amperes; maximum load, 25 amperes; appliances not in excess of 20 amperes.

d. 35-*Ampere Branch Circuit.* Wire not smaller than No. 8; protection, 35 amperes; maximum load, 35 amperes; for use in other than dwelling occupancy.

e. 50-*Ampere Branch Circuit.* Wire not smaller than No. 6; protection, 50 amperes; maximum load, 50 amperes, for use in other than dwelling occupancy.

f. *Motor Branch Circuits.* Separate treatment for each problem.

18. Motor Branch Circuit. The motor branch circuit is so designed that the wire will have a current-carrying capacity of 25 per cent more than the motor rating, to provide for the starting current of the motor. Tables V–14 give the permissible current-carrying capacity of copper wire with the various types of insulation, and Tables VI–14 give the allowable number of wires of various sizes in different sizes of conduit. If various sizes of wire are placed in the same conduit, the size of the conduit is governed by the overall cross sections of the conductors. The permissible areas that can be occupied by the wire are given in Table VI–14*E*. Tables VII–14 give the proper size fuse and circuit breaker setting to use with motor installations. Tables VIII–14 to XII–14 give current values and power factors to be applied to the solution of motor problems.

TABLE V-14A [1]

ALLOWABLE CURRENT-CARRYING CAPACITIES OF CONDUCTORS IN AMPERES
Not More Than Three [2,3] Conductors in Raceway or Cable
(Based on Room Temperature of 30° C, 86° F)

Size AWG MCM	Rubber Type RW Type R	Synthetic Type SN / Type RU / Rubber Type RPT Type RP	Rubber Type RHT Type RH	Paper / Synthetic Type SNA / Asbestos Var-Cam Type AVB / Var-Cam Type V	Asbestos Var-Cam Type AVA Type AVL	Impregnated Asbestos Type AI	Asbestos Type A
14	15	18	22	23	28	29	32
12	20	23	27	29	36	38	42
10	25	31	37	38	47	49	54
8	35	41	49	50	60	63	71
6	45	54	65	68	80	85	95
5	52	63	75	78	94	99	110
4	60	72	86	88	107	114	122
3	69	83	99	104	121	131	145
2	80	96	115	118	137	147	163
1	91	110	131	138	161	172	188
0	105	127	151	157	190	202	223
00	120	145	173	184	217	230	249
000	138	166	199	209	243	265	284
0000	160	193	230	237	275	308	340
250	177	213	255	272	315	334	372
300	198	238	285	299	347	380	415
350	216	260	311	325	392	419	462
400	233	281	336	361	418	450	488
500	265	319	382	404	468	498	554
600	293	353	422	453	525	543	612
700	320	385	461	488	562	598	668
750	330	398	475	502	582	621	690
800	340	410	490	514	600	641	720
900	360	434	519	556			
1000	377	455	543	583	681	730	811
1250	409	493	589	643			
1500	434	522	625	698	784		
1750	451	544	650	733			
2000	463	558	666	774	839		

CORRECTION FACTOR FOR ROOM TEMPERATURES OVER 30° C

C	F							
40	104	0.71	0 82	0.88	0.90	0.94	0.95	
45	113	0 50	0.71	0.82				
50	122	0.00	0.58	0.75	0.80	0.87	0.89	
55	131		0.41	0.67				
60	140		0.00	0.58	0.67	0.79	0.83	0.97
70	158			0.35	0.52	0.71	0.76	0.93
75	167			0.00				
80	176				0.30	0.61	0.69	0.89
90	194					0.50	0.61	0.86
100	212						0.51	0.82
120	248							0.72
140	284							0.63

[1] National Electrical Code, 1940.
[2] For 4 to 6 wires use 80 per cent of current rating.
[3] For 7 to 9 wires use 70 per cent of current rating.

TABLE V-14B[1]

Allowable Current-Carrying Capacities of Conductors in Amperes
Single Conductor in Free Air
(Based on Room Temperature of 30° C, 86° F)

Size AWG or MCM	Rubber Type R	Rubber Type RP	Rubber Type RHT Type RH	Synthetic Type SNA / Asbestos Var-Cam Type AVB / Var-Cam Type V	Asbestos Var-Cam Type AVA	Impregnated Asbestos Type AI	Asbestos Type A	Slow-Burning Type SB / Weatherproof Type W Type SBW
14	20	24	29	30	39	40	43	23
12	26	31	37	40	51	52	57	30
10	35	42	50	54	65	69	75	40
8	48	58	69	71	85	91	100	53
6	65	78	94	99	119	126	134	70
5	76	92	110	115	136	145	158	80
4	87	105	125	133	158	169	180	90
3	101	122	146	155	182	194	211	100
2	118	142	170	179	211	226	241	125
1	136	164	196	211	247	264	280	150
0	160	193	230	245	287	306	325	200
00	185	223	267	284	331	354	372	225
000	215	259	310	330	384	410	429	275
0000	248	298	358	383	446	476	510	325
250	280	338	403	427	495	528	562	350
300	310	373	446	480	555	592	632	400
350	350	421	504	529	612	653	698	450
400	380	457	547	575	665	710	755	500
500	430	517	620	660	765	814	870	600
600	480	577	691	738	857	912	970	680
700	525	632	756	813	942	1003	1065	760
750	545	655	785	846	981	1044	1118	800
800	565	680	815	879	1020	1085	1150	840
900	605	728	872	941				920
1000	650	782	936	1001	1163	1238	1332	1000
1250	740	890	1066	1131				
1500	815	980	1174	1261	1452			1360
1750	890	1070	1282	1370				
2000	960	1155	1383	1472	1713			1670

CORRECTION FACTOR FOR ROOM TEMPERATURES OVER 30° C

C F								
40 104	0.71	0.82	0.88	0.90	0.94	0.95		
45 113	0.50	0.71	0.82					
50 122	0.00	0.58	0.75	0.80	0.87	0.89		
55 131		0.41	0.67					
60 140		0.00	0.58	0.67	0.79	0.83	0.97	
70 158			0.35	0.52	0.71	0.76	0.93	
75 167			0.00					
80 176				0.30	0.61	0.69	0.89	
90 194					0.50	0.61	0.86	
100 212						0.51	0.82	
120 248							0.72	
140 284							0.63	

[1] National Electrical Code, 1940.

TABLE V-14C[1]

PROPERTIES OF CONDUCTORS

AWG	CM	Ohms per 1000 ft 15° C–59° C	Bare Conductor		Concentric Lay Stranded Conductors. Rubber, Paper, Asbestos, Varnished Cambric, Asbestos Varnished Cambric	
			Diameter Inches	Area Square Inches	No. of Wires	Diameter Each Wire Inches
14	4,107	2.475	0.064	0.003	7	0.024
12	6,530	1.557	0.081	0.005	7	0.030
10	10,380	0.9792	0.102	0.008	7	0.038
8	16,510	0.6158	0.128	0.013	7	0.048
6	26,250	0.3872	0.184	0.026	7	0.061
5	33,100	0.3071	0.213	0.035	7	0.068
4	41,740	0.2436	0.232	0.042	7	0.077
3	52,630	0.1961	0.261	0.053	7	0.086
2	66,370	0.1532	0.292	0.067	7	0.097
1	83,690	0.1215	0.332	0.087	19	0.066
0	105,500	0.09633	0.375	0.110	19	0.074
00	133,100	0.07639	0.419	0.138	19	0.083
000	167,800	0.06058	0.470	0.173	19	0.094
0000	211,600	0.04804	0.528	0.219	19	0.105
	250,000	0.04147	0.594	0.276	37	0.082
	300,000	0.03457	0.641	0.323	37	0.090
	350,000	0.02963	0.688	0.370	37	0.097
	400,000	0.02592	0.734	0.423	37	0.104
	500,000	0.02074	0.828	0.540	37	0.116
	600,000	0.01729	0.892	0.628	61	0.099
	700,000	0.01481	0.968	0.735	61	0.107
	750,000	0.01382	1.000	0.785	61	0.110
	800,000	0.01296	1.031	0.835	61	0.114
	900,000	0.01153	1.094	0.938	61	0.121
	1,000,000	0.01036	1.172	1.039	61	0.128
	1,250,000	0.00829	1.290	1.320	91	0.117
	1,500,000	0.00692	1.422	1.580	91	0.128
	1,750,000	0.00593	1 546	1.872	127	0.117
	2,000,000	0.00518	1.630	2.084	127	0.125

[1] National Electrical Code, 1940.

TABLE VI- 4A

NUMBER OF CONDUCTORS IN CONDUIT OR TUBING

Size of Wire	Number of Wires in One Conduit: 1	2	3	4	5	6	7	8	9
	Minimum Size of Conduit in Inches								
No. 14	½	½	½	½	¾	¾	¾	1	1
12	½	½	½	¾	¾	1	1	1	1¼
10	½	¾	¾	¾	1	1	1¼	1¼	1¼
8	½	¾	¾	1	1	1¼	1¼	1¼	1¼
6	½	1	1¼	1¼	1½	1½	2	2	2
5	¾	1¼	1¼	1¼	1½	2	2	2	2
4	¾	1¼	1¼	1½	2	2	2	2	2½
3	¾	1¼	1¼	1½	2	2	2	2½	2½
2	¾	1¼	1½	1½	2	2	2½	2½	2½
1	¾	1½	1½	2	2	2½	2½	3	3
0	1	1½	2	2	2½	2½	3	3	3
00	1	2	2	2½	2½	3	3	3	3½
000	1	2	2	2½	3	3	3	3½	3½
0,000	1¼	2	2½	2½	3	3	3½	3½	4
200,000 CM	1¼	2	2½	2½	3	3	3½	3½	4
225,000	1¼	2½	2½	3	3	3½			
250,000	1¼	2½	2½	3	3	3½			
300,000	1¼	2½	3	3	3½	3½			
350,000	1¼	2½	3	3½	3½	4			
400,000	1¼	3	3	3½	4	4			
450,000	1½	3	3	3½	4	4½			
500,000	1½	3	3	3½	4	4½			
550,000	1½	3	3½	4	4½	5			
600,000	2	3	3½	4	4½	5			
650,000	2	3½	3½	4					
700,000	2	3½	3½	4½					
750,000	2	3½	3½	4½					
800,000	2	3½	4	4½					
850,000	2	3½	4	4½					
900,000	2	3½	4	4½					
950,000	2	4	4	5					
1,000,000	2	4	4	5					
1,100,000	2½	4	4½	6					
1,200,000	2½	4½	4½	6					
1,250,000	2½	4½	4½	6					
1,300,000	2½	4½	5	6					
1,400,000	2½	4½	5	6					
1 500,000	2½	4½	5	6					
1,600,000	2½	5	5	6					
1,700,000	3	5	5	6					
1,750,000	3	5	5	6					
1,800,000	3	5	6	6					
1,900,000	3	5	6						
2,000,000	3	5	6						

[1] National Electrical Code, 1940.

LEAD COVERED WIRES AND CABLES—600 VOLTS

Size of Conduit in Inches to Contain Not More than Four Cables

Size of Wire	Single Conductor Cable: 1	2	3	4	2-Conductor Cable: 1	2	3	4	3-Conductor Cable: 1	2	3	4
	Cables in One Conduit				Cables in One Conduit				Cables in One Conduit			
14	½	¾	¾	¾	½	1	1	1¼	¾	1¼	1½	1½
12	½	¾	¾	1	¾	1	1¼	1¼	1	1¼	1½	2
10	½	¾	1	1	¾	1¼	1¼	1½	1	1½	2	2
8	½	1	1	1¼	1	1¼	1½	2	1	2	2	2½
6	¾	1¼	1¼	1½	1¼	1½	2	2½	1¼	2½	3	3
4	¾	1¼	1½	1½	1¼	2	2½	2½	1½	3	3	3½
3	¾	1¼	1½	2	1¼	2	2½	3	1½	3	3	3½
2	1	1¼	1½	2	1¼	2	2½	3	1½	3	3½	4
1	1	1½	2	2	1½	2½	3	3½	2	3½	4	4½
0	1	2	2	2½	2	2½	3	3½	2	4	4½	5
00	1	2	2	2½	2	3	3½	4	2½	4	4½	5
000	1¼	2	2½	2½	2	3	3½	4	2½	4½	4½	6
0 000	1¼	2½	2½	3	2½	3	3½	4½	3	5	6	6
250,000	1¼	2½	3	3					3	6	6	
300 000	1½	3	3	3½					3½	6	6	
350,000	1½	3	3	3½					3½	6	6	
400 000	1½	3	3	3½					3½	6	6	
450 000	1½	3	3	4					4	6	6	
500 000	1½	3	3½	4					4	6		
600,000	2	3½	4	4½								
700,000	2	4	4	5								
750,000	2	4	4	5								
800,000	2	4	4½	5								
900 000	2½	4	4½	5								
1,000 000	2½	4½	4½	6								...
1,250 000	3	5	5	6								...
1,500,000	3	5	6	6								...
1,750,000	3	6	6									...
2,000,000	3½	6	6									...

TABLE VI-14B[1]

NUMBER OF CONDUCTORS IN CONDUIT OR TUBING

Small Diameter Building Wire, Types RHT and RPT, 600 Volts. One to Nine Conductors

(Nominal diameter in inches)

Size of Conductor	Number of Conductors in One Conduit or Tubing								
	1	2	3	4	5	6	7	8	9
14	½	½	½	½	½	½	½	¾	¾
12	½	½	½	½	½	¾	¾	¾	¾
10	½	½	½	½	¾	¾	¾	¾	1
8	½	½	¾	¾	1	1	1	1	1¼

[1] National Electrical Code, 1940.

TABLE VI-14C[1]

NUMBER OF CONDUCTORS IN CONDUIT OR TUBING

Synthetic, Type SN and Type RU, 600 Volts. One to Nine Conductors

(Nominal diameter in inches)

Size of Conductor	Number of Conductors in One Conduit or Tubing								
	1	2	3	4	5	6	7	8	9
14	½	½	½	½	½	½	½	½	½
12	½	½	½	½	½	½	½	½	¾
10	½	½	½	½	½	½	¾	¾	¾
8	½	½	½	¾	¾	¾	1	1	1
6	½	¾	¾	1	1	1¼	1¼	1¼	1¼
5	½	¾	¾	1	1¼	1¼	1¼	1¼	1½
4	½	¾	1	1	1¼	1¼	1¼	1½	1½
3	½	1	1	1¼	1¼	1¼	1½	1½	2
2	½	1	1	1¼	1¼	1½	1½	2	2
1	¾	1¼	1¼	1½	1½	2	2	2	2½
0	¾	1¼	1¼	1½	2	2	2	2½	2½
00	¾	1¼	1½	2	2	2	2½	2½	2½
000	¾	1¼	1½	2	2	2½	2½	3	3
0000	1	1½	2	2	2½	2½	3	3	3

[1] National Electrical Code, 1940.

TABLE VI-14D[1]

NUMBER OF CONDUCTORS IN CONDUIT OR TUBING

More than Nine Conductors, Rubber-Covered Types R, RW, RP, and RH—600 Volts

Size of Conductor	Maximum Number of Conductors in Conduit or Tubing						
	Inches						
	¾	1	1¼	1½	2	2½	3
18	13	22	38	53	87	124	191
16	11	19	33	45	74	106	163
14		11	19	26	43	61	95
12			15	21	34	50	77
10			12	16	27	38	60
8				13	22	31	49
6						14	22

[1] National Electrical Code, 1940.

TABLE VI-14E[1]

CONDUIT FOR COMBINATION OF CONDUCTORS

Per Cent Area of Conduit or Tubing

	Number of Conductors				
	1	2	3	4	Over 4
Conductors (not lead covered)	53	31	43	40	40
Lead-covered conductors	55	30	40	38	35
For rewiring existing raceways with thinner insulated conductors	60	40	50	50	50

[1] National Electrical Code, 1940.

TABLE VI–14F[1]

CONDUIT FOR COMBINATION OF CONDUCTORS

Dimensions of Rubber-Covered Conductors

(Types R, RW, RP, and RH)

Size AWG–CM	Approx. Diameter Inches	Approx. Area Sq In.	Size CM	Approx. Diameter Inches	Approx. Area Sq In.
18	0.14	0.0154	450,000	1.08	0.91
16	0.15	0.018	500,000	1.12	0.99
14	0.20	0.031	550,000	1.17	1.08
12	0.22	0.038	600,000	1.22	1.16
10	0.24	0.045			
8	0.30	0.071	650,000	1.25	1.23
			700,000	1.29	1.30
6	0.41	0.13	750,000	1.33	1.38
4	0.45	0.16	800,000	1.36	1.45
2	0.52	0.21			
1	0.59	0.27	850,000	1.39	1.52
			900,000	1.43	1.60
0	0.63	0.31	950,000	1.46	1.68
00	0.67	0.35	1,000,000	1.49	1.75
000	0.72	0.41			
0000	0.78	0.48	1,250,000	1.68	2.22
			1,500,000	1.79	2.52
250,000	0.86	0.58	1,750,000	1.90	2.85
300,000	0.92	0.67	2,000,000	2.00	3.14
350,000	0.98	0.75			
400,000	1.03	0.83			

[1] National Electrical Code, 1940.

TABLE VI–14G[1]

CONDUIT FOR COMBINATION OF CONDUCTORS

Dimensions of Conductors

(Small-diameter building wires, types RHT and RPT)

Size AWG	Approx. Diameter Inches	Approx. Area Sq In.	Size AWG	Approx. Diameter Inches	Approx. Area Sq In.
14	0.162	0.0206	10	0.200	0.0314
12	0.179	0.0252	8	0.261	0.0535

[1] National Electrical Code, 1940.

TABLE VI-14H [1]

CONDUIT FOR COMBINATION OF CONDUCTORS

Dimension of Conductors

(Synthetic insulation, type SN, type RU insulation)

Size AWG	Approx. Diameter Inches	Approx. Area Sq In.	Size AWG	Approx. Diameter Inches	Approx. Area Sq In.
14	0.130	0.0133	2	0.423	0.1405
12	0.147	0.0170	1	0.496	0.1935
10	0.168	0.0220	0	0.537	0.226
8	0.227	0.0405	00	0.583	0.267
6	0.314	0.0775	000	0.634	0.316
4	0.363	0.1035	0000	0.692	0.376

[1] National Electrical Code, 1940.

TABLE VI-14I [1]

CONDUIT FOR COMBINATION OF CONDUCTORS

Dimensions of Lead-Covered Conductors

(Types RL, RPL, and RHL)

Size of Conductor AWG–CM	Single-Conductor		Two-Conductor		Three-Conductor	
	Diam. Inches	Area Sq In.	Diameter Inches	Area Sq In.	Diam. Inches	Area Sq In.
14	0.28	0.062	0.28 × 0.47	0.115	0.59	0.273
12	0.29	0.066	0.31 × 0.54	0.146	0.62	0.301
10	0.35	0.096	0.35 × 0.59	0.180	0.68	0.363
8	0.41	0.132	0.41 × 0.71	0.255	0.82	0.528
6	0.49	0.188	0.49 × 0.86	0.369	0.97	0.738
4	0.55	0.237	0.54 × 0.96	0.457	1.08	0.916
2	0.60	0.283	0.61 × 1.08	0.578	1.21	1.146
1	0.67	0.352	0.70 × 1.23	0.756	1.38	1.49
0	0.71	0.396	0.74 × 1.32	0.859	1.47	1.70
00	0.76	0.454	0.79 × 1.41	0.980	1.57	1.94
000	0.81	0.515	0.84 × 1.52	1.123	1.69	2.24
0000	0.87	0.593	0.90 × 1.64	1.302	1.85	2.68
250	0.98	0.754			2.02	3.20
300	1.04	0.85			2.15	3.62
350	1.10	0.95			2.26	4.02
400	1.14	1.02			2.40	4.52
500	1.23	1.18			2.59	5.28

[1] National Electrical Code, 1940.

TABLE VI–14J[1]

CONDUIT FOR COMBINATION OF CONDUCTORS

Dimensions of Asbestos-Varnished-Cambric Insulated Conductors (Types AVA, AVB, and AVL)

Size AWG CM	Type AVA		Type AVB		Type AVL	
	Approx. Diam. Inches	Approx. Area Sq In.	Approx. Diam. Inches	Approx. Area Sq In.	Approx. Diam. Inches	Approx. Area Sq In.
14	0.245	0.047	0.205	0.033	0.320	0.080
12	0.265	0.055	0.225	0.040	0.340	0.091
10	0.285	0.064	0.245	0.047	0.360	0.102
8	0.310	0.075	0.270	0.057	0.390	0.119
6	0.395	0.122	0.345	0.094	0.430	0.145
4	0.445	0.155	0.395	0.123	0.480	0.181
2	0.505	0.200	0.460	0.166	0.570	0.255
1	0.585	0.268	0.540	0.229	0.620	0.300
0	0.625	0.307	0.580	0.264	0.660	0.341
00	0.670	0.353	0.625	0.307	0.705	0.390
000	0.720	0.406	0.675	0.358	0.755	0.447
0000	0.780	0.478	0.735	0.425	0.815	0.521
250,000	0.885	0.616	0.855	0.572	0.955	0.715
300,000	0.940	0.692	0.910	0.649	1.010	0.800
350,000	0.995	0.778	0.965	0.731	1.060	0.885
400,000	1.040	0.850	1.010	0.800	1.105	0.960
450,000	1.085	0.925	1.055	0.872	1.150	1.040
500,000	1.125	0.995	1.095	0.945	1.190	1.118
550,000	1.165	1.065	1.135	1.01	1.265	1.26
600,000	1.205	1.140	1.175	1.09	1.305	1.34
650,000	1.240	1.21	1.210	1.15	1.340	1.41
700,000	1.275	1.28	1.245	1.22	1.375	1.49
750,000	1.310	1.35	1.280	1.29	1.410	1.57
800,000	1.345	1.42	1.315	1.36	1.440	1.63
850,000	1.375	1.49	1.345	1.43	1.470	1.70
900,000	1.405	1.55	1.375	1.49	1.505	1.78
950,000	1.435	1.62	1.405	1.55	1.535	1.85
1,000,000	1.465	1.69	1.435	1.62	1.565	1.93

[1] National Electrical Code, 1940.

TABLE VI-14K [1]

CONDUIT FOR COMBINATION OF CONDUCTORS

Dimensions of Conduit or Tubing

Size	Internal Diameter Inches	Area Square Inches	Size	Internal Diameter Inches	Area Square Inches
½	0.622	0.30	3	3.068	7.38
¾	0.824	0.53	3½	3.548	9.90
1	1.049	0.86	4	4.026	12.72
1¼	1.380	1.50	4½	4.506	15.95
1½	1.610	2.04	5	5.047	20.00
2	2.067	3.36	6	6.065	28.89
2½	2.469	4.79			

[1] National Electrical Code, 1940.

TABLE VII-14A [1]

MAXIMUM RATING OR SETTING OF MOTOR-BRANCH-CIRCUIT PROTECTIVE DEVICES FOR MOTORS NOT MARKED WITH A CODE LETTER INDICATING LOCKED ROTOR KVA

Type of Motor	Per Cent of Full-Load Current		
	Fuse Rating	Circuit Breaker Setting	
		Instantaneous Type	Time-Limit Type
Single-phase, all types	300		250
Squirrel cage and synchronous (full voltage, resistor, and reactor starting)	300		250
Squirrel cage and synchronous (autotransformer starting):			
Not more than 30 amp	250		200
More than 30 amp	200		200
High reactance squirrel cage:			
Not more than 30 amp	250		250
More than 30 amp	200		200
Wound-rotor	150		150
D-c:			
Not more than 50 hp	150	250	150
More than 50 hp	150	175	150

Synchronous motors of the low torque, low speed type (usually 450 rpm or lower), such as are used to drive reciprocating compressors, and pumps, which start up unloaded, do not require a fuse rating or circuit breaker setting in excess of 200 per cent of full load current.

[1] National Electrical Code, 1940.

TABLE VII-14B [1]

MAXIMUM RATING OR SETTING OF MOTOR-BRANCH-CIRCUIT PROTECTIVE DEVICES FOR MOTORS MARKED WITH A CODE LETTER INDICATING LOCKED ROTOR KVA

Type of Motor	Per Cent of Full-Load Current		
	Fuse Rating	Circuit Breaker Setting	
		Instantaneous Type	Time-Limit Type
All a-c single-phase and polyphase squirrel cage and synchronous motors with full voltage, resistor, or reactor starting:			
Code letter A	150		150
Code letter B to E	250		200
Code letter F to R	300		250
All a-c squirrel cage and synchronous motors with autotransformer starting:			
Code letter A	150		150
Code letter B to E	200		200
Code letter F to R	250		200

Synchronous motors of the low torque, low speed type (usually 450 rpm or lower), such as are used to drive reciprocating compressors, and pumps, which start up unloaded, do not require a fuse rating or circuit breaker setting in excess of 200 per cent of full load current.

[1] National Electrical Code, 1940.

TABLE VIII-14 [1]

Full-Load Motor Currents [2]—D-C Motors—Amperes

Hp	115 Volts	230 Volts	550 Volts
½	4.5	2.3	.
¾	6.5	3.3	1.4
1	8.4	4.2	1.7
1½	12.5	6.3	2.6
2	16.1	8.3	3.4
3	23.0	12.3	5.0
5	40	19.8	8.2
7½	58	28.7	12.0
10	75	38	16.0
15	112	56	23.0
20	140	74	30
25	185	92	38
30	220	110	45
40	294	146	61
50	364	180	75
60	436	215	90
75	540	268	111
100	.....	357	146
125	.....	443	184
150	.....	.....	220
200	.....	.	295

[1] National Electrical Code, 1940.

[2] These values of full-load current are average for all speeds.

TABLE IX-14 [1]

Full-Load Motor Currents—Single-Phase A-C Motors—Amperes

Hp	110 Volts	220 Volts	440 Volts
⅙	3.34	1.67	
¼	4.8	2.4	...
½	7	3.5	. ..
¾	9.4	4.7	.
1	11	5.5	. .
1½	15.2	7.6	....
2	20	10	...
3	28	14	..
5	46	23	....
7½	68	34	17
10	86	43	21.5

For full-load currents of 208- and 200-volt motors, increase corresponding 220-volt motor full-load current by 6 and 10 per cent, respectively.

[1] National Electrical Code, 1940.

TABLE X-14

Equipment Power Factors

Equipment	Power Factor
Incandescent lamps	100%
Heating equipment	100
Arc lamps	60
Low pressure mercury vapor lamps (old style)	85
Low pressure mercury vapor lamps (new style)	90
High pressure mercury vapor lamps (uncorrected)	50–70
High pressure mercury vapor lamps (corrected)	94
Fluorescent lamps (uncorrected)	60
Fluorescent lamps (corrected)	95
Motors (full load)	
1 hp	70
2	75
3	80.5
5	82
7½	82.5
10	84
15	86
20	88
25	88
30	88.5
40	89
50	89
60	90
75	90.5
100	91
125	92.5
150	94
200	96
Motors starting	50

TABLE XI-14[1, 2]

Two-Phase A-C Motors (4-Wire)

Full-Load Current

	Induction Type Squirrel Cage and Wound Rotor Amperes					Synchronous Type Unity Power Factor Amperes			
Hp	110 volts	220 volts	440 volts	550 volts	2200 volts	220 volts	440 volts	550 volts	2200 volts
½	4.3	2.2	1.1	.9		Values for unity pf 80% × 1.25 90% × 1.1			
¾	4.7	2.4	1.2	1.0					
1	5.7	2.9	1.4	1.2	.				
1½	7.7	4.0	2	1.6	..				
2	10.4	5	3	2.0					
3	.	8	4	3.0	.				
5	...	13	7	6	..				
7½	.	19	9	7					
10		24	12	10					
15	..	33	16	13	..				
20	.	45	23	19	.				
25	.	55	28	22	6	47	24	19	4.7
30	..	67	34	27	7	56	29	23	5.7
40	..	88	44	35	9	75	37	31	7.5
50		108	54	43	11	94	47	38	9.4
60	.	129	65	52	13	111	56	44	11.3
75		156	78	62	16	140	70	57	14
100		212	106	85	22	182	93	74	18
125	.	268	134	108	27	228	114	93	23
150		311	155	124	31		137	110	28
200	..	415	208	166	43		182	145	37

[1] National Electrical Code, 1940.
[2] Common wire 1.41 times values given.

TABLE XII-14[1]

Three-Phase A-C Motors

Full-Load Current

	Induction Type Squirrel Cage and Wound Rotor Amperes					Synchronous Type Unity Power Factor Amperes			
Hp	110 volts	220 volts	440 volts	550 volts	2200 volts	220 volts	440 volts	550 volts	2200 volts
½	5	2.5	1.3	1		Unity pf 90% × 1.1 80% × 1.25			
¾	5.4	2.8	1.4	1.1					
1	6.6	3.3	1.7	1.3					
1½	9.4	4.7	2.4	2.0					
2	12	6	3	2.4					
3		9	4.5	4					
5	.	15	7.5	6					
7½	.	22	11	9					
10		27	14	11					
15	.	38	19	15					
20	.	52	26	21					
25		64	32	26	7	54	27	22	5.4
30		77	39	31	8	65	33	26	6.5
40		101	51	40	10	86	43	35	8.6
50		125	63	50	13	108	54	44	10.8
60	....	149	75	60	15	128	64	51	13
75	....	180	90	72	19	161	81	65	16
100	...	246	123	98	25	211	106	85	21
125	..	310	155	124	32	164	132	106	26
150	.	360	180	144	36	.	158	127	32
200	.	480	240	195	49	.	210	168	42

For full-load currents of 208- and 200-volt motors, increase the corresponding 220-volt motor full-load current by 6 and 10%, respectively.

[1] National Electrical Code, 1940.

19. Feeders. The determination of the feeder size for either lighting or power loads, or their combination, depends upón the power required by the branch circuit system. Frequently the size of the feeder may be reduced by the consideration of a probable demand factor. For values of demand an electrical engineering handbook should be consulted and the final installation must be approved by the authority enforcing the code. The solution without the demand factor will give the highest safety factor.

For the lighting feeder, it is necessary only to determine the total load on the connected branches and to select a wire size from Table V–14*A* or V–14*B* which will carry the current safely. The fuse protection will be the limiting current-carrying capacity of the wire.

For the power feeder, it is necessary to design the wire so that the largest branch circuit current, plus the full load current of the other branches will be the carrying capacity of the wire selected from Table V–14*A* or V–14*B*. The branch calculation requires 25 per cent over capacity for the largest branch. The fuse or circuit breaker is specified, as is the feeder wire size, by taking care of the requirements of the largest branch circuit and then adding the full load current of the other branches.

Example e. Design the branch circuit wire and fuse protection for a 30-hp, 230-volt, d-c motor. The branch is 150 ft long and, in open wiring, spaced on 4-in. centers using Type RP rubber-covered copper wire for the conductor. Determine the allowable wire size and the voltage drop.

From Table VIII–14, $I = 110$ amp

$$110 \times 1.25 = 138 \text{ amp}$$

From Table V–14*B*, Use wire No. 2

From Table VII–14*A*, Fuse 175 amp (110 × 1.5)

$$VD = \frac{11 \times 110 \times 150 \times 2}{66{,}370} = 5.47 \text{ volts}$$

$$\frac{5.47}{230} \times 100 = 2.4 \text{ per cent}$$

Example f. Design the branch circuit and branch circuit protection for a 30-hp, 3ϕ, 220-volt, 60-cycle, squirrel cage induction motor, using an autotransformer for starting. The branch circuit is 150 ft long and, in open wiring, spaced 4 in. apart, using Type RP rubber-covered copper wire. Determine the allowable wire size and the voltage drop.

From Table XII–14, $I = 77$ amp

$$77 \times 1.25 = 96.3 \text{ amp}$$

From Table V–14*B*,	Use wire No. 4
From Table VII–14*A*,	Fuse 175 amp (77 × 2.0)

$$VD = \frac{11 \times 77 \times 150 \times 1 \times 1.1 \times 1.73}{41{,}740} = 5.78 \text{ volts}$$

From Table IV–14,	$K_1 = 1 \qquad K_3 = 1.73$
From curve, Fig. 22–14, $K_2 = 1.1$	Effective spacing 6 in. Power factor 88.5

$$\frac{5.78}{220} = 2.6 \text{ per cent}$$

Example g. A 2 per cent voltage drop is specified on a motor branch circuit supplying a 100-hp, three-phase, 25-cycle, 440-volt squirrel cage, induction motor with full voltage reactance starting. The run is 400 ft and is installed in conduit, Type RP rubber-covered copper wire being used. Design the wire size, the fuse protection, and specify the conduit size.

Table XII–14,	$I = 123$ amp
Allowable drop, 440 × 0.02 = 8.8 volts	
Table X–14,	Power factor 91 per cent

$$123 \times 1.25 = 154 \text{ amp}$$

Table V–14*A*	Use No. 000 wire

Wire No. 000 satisfies the safety requirements, but does it satisfy the voltage requirements?

$$CM = \frac{11 \times 123 \times 400 \times 1 \times K_2 \times 1.73}{440 \times 0.02}$$

Since the factor K_2 depends upon the circular mil size of the wire, the problem becomes one of "trial and error." Determine the ratio of CM/K_2 and select a wire that will be satisfactory.

$$\frac{CM}{K_2} = 106{,}200$$

The value of CM/K_2 for No. 000 wire will be

$$\frac{CM}{K_2} = \frac{167{,}800}{1.06} = 158{,}000$$

$$K_2 \begin{Bmatrix} \text{Conduit} \\ \text{No. 000} \\ \text{pf} = 91 \text{ per cent} \end{Bmatrix} \begin{matrix} \\ \text{Fig. 23–14,} \\ K_2 = 1.06 \end{matrix}$$

The required safe wire will not give an excessive voltage drop.

Summary: 3—No. 000 wires
3—400-amp fuses (Table VII–14*A*)
2—inch conduit (Table VI–14*A*)

Example h. Panels supplying 60 branch circuits of 10- to 100-watt lamps each require a feeder having a drop not in excess of 1.5 per cent if installed in conduit with Type RP rubber-covered copper wire. The panel is located 200 ft. from the supply.

If the installation is for direct current, determine (*a*) the wire size for a 2-wire 110-volt system and (*b*) the wire size for a 3-wire 110/220-volt system.

Two-wire system:

$$\frac{10 \times 100 \times 60}{110} = 545 \text{ amp}$$

$$CM = \frac{11 \times 545 \times 200 \times 2}{110 \times 0.015} = 1{,}450{,}000 \text{ cir mils (not safe)}$$

This is an impractical size of wire; therefore, the 3-wire supply must be used.

Three-wire system:

$$\frac{10 \times 100 \times 60}{220} = 273 \text{ amp}$$

$$CM = \frac{11 \times 273 \times 200 \times 1}{110 \times 0.015} = 364{,}000$$

From Table V–14*A*, 400,000 cir mils would be safe.

The design requires 400,000 cir mils with a 300-amp fuse to obtain proper voltage drop.

Summary: 3—400,000 cir mils
2—300-amp fuses (Table V–14*A*)
3—inch conduit (Table VI–14*A*)

The load should be divided into two feeders.

Example i. Design the wire size, fuse size, and conduit size and determine the voltage on the lamps under the following specifications.

Service panels for 60 lighting branch circuits, each branch servicing 10- to 100-watt lamps. Three-wire, 60-cycle, 110/220 volt service. The supply voltage varies from 217 to 223 volts, Type RP rubber-covered copper wire and 200-ft feeder. Maximum voltage drop 3 per cent.

3 per cent drop: $110 \times 0.03 = 3.3$ volts.

Supply drop: $\frac{217}{2} = 108.5 \qquad 110 - 108.5 = 1.5$ volts

Allowable feeder drop: $3.3 - 1.5 = 1.8$ volts

$$I = \frac{60 \times 10 \times 100}{220} = 273 \text{ amp}$$

$$CM = \frac{11 \times 273 \times 200 \times 1 \times K_2 \times 1}{1.8}$$

$$\frac{CM}{K_2} = 334{,}000$$

By "trial and error" the proper wire must be selected so the ratio of CM/K_2 will be satisfied.

400,000 cir mils is the safe wire (Table V–14A).

$$\frac{CM}{K_2} = \frac{400{,}000}{1.07} = 374{,}000$$

$$K_2 \left\{ \begin{matrix} \text{Conduit} \\ 400{,}000 \text{ cir mils} \\ \text{pf} = 1 \end{matrix} \right\} \text{ From Fig. 22–14} \qquad K_2 = 1.07$$

Therefore the 400,000-cir-mil wire is satisfactory.

Summary: 3—400,000 cir mils
2—300-amp fuses (Table V–14A)
3—inch conduit (Table VI–14A)
106.7 volts—lowest voltage on lamps when system is loaded

Example j. Design the branch circuits and feeder for the following 230-volt, d-c motor supply. The wire used is Type RP rubber-covered copper spaced on 4-in. centers for feeder; branches are in conduit.

Branch A:	25-hp, 230-volt d-c motor	150 ft
Branch B:	50-hp, 230-volt d-c motor	100 ft
Feeder:	230-volt d-c 2-wire	200 ft

Branch	Full-Load Current	Starting Current	Wire	Fuse	VD
A	92	·115	No. 0	150	2.88
B	180	225	300,000/CM	300	1.32

$$VD_A = \frac{11 \times 92 \times 150 \times 2}{105{,}500} = 2.88 \text{ volts}$$

$$VD_B = \frac{11 \times 180 \times 100 \times 2}{300{,}000} = 1.32 \text{ volts}$$

Feeder:

Full-Load Current	Starting Current	Wire	Fuse	VD
272	317	250,000	400	4.8

$$VD = \frac{11 \times 272 \times 200 \times 2}{250{,}000} = 4.8 \text{ (2.1 per cent)}$$

The voltage drop to motor A will be 7.7 volts.
The voltage drop to motor B will be 6.1 volts.

The allowable drop for motor service is 10 per cent or 23 volts on the 230-volt system. This installation satisfies both the safety and adequacy conditions.

For the combined lighting and power feeder, it is necessary only to treat the lighting load as in lighting feeders, and the power load as in power feeders. The two are then added and the wire size is determined from Table V–14.

In each problem, it has been observed that, if the feeder is treated as an installed wire, the problem is simple; but, if the feeder is considered

from the standpoint of design, it is necessary, to use the method of "trial and error," that is, if voltage drop is to be considered. When voltage drop is to be neglected, the design is merely the selection of the minimum wire size permitted by the National Electrical Code or the rules operative in the community. The selection of safe wire size is a simple problem, but the selection of an adequate wire for delivering the proper voltage to the equipment is a problem for the engineer.

Example k. Design the safe branch circuits and feeder for the following three-phase, 25-cycle, 220-volt, a-c system. The wire used is Type RP rubber-covered copper spaced on 4-in. centers.

Branch A:	25-hp, 220-volt, squirrel cage induction motor, autotransformer starting	150 ft
Branch B:	50-hp, 220-volt, synchronous motor, reactance starting and operating at 90 per cent power factor	100 ft
Feeder:	3-wire, 220-volt, 25-cycle	200 ft

Branch	Full-Load Current	Starting Current	PF Per Cent	Wire	Fuse	*VD*
A	64	80	88	No. 5	150	5.5
B	119	149	90	No. 1	400	2.9
	183					

$$VD_A = \frac{11 \times 64 \times 150 \times 1 \times 1 \times 1.73}{33{,}100} = 5.53 \text{ volts (2.5 per cent)}$$

$$K_2 \left\{ \begin{matrix} \text{6-in. spacing} \\ \text{pf} = 88 \\ \text{No. 5} \end{matrix} \right\} \text{From Fig. 23–14} \qquad K_2 = 1$$

$$VD_B = \frac{11 \times 119 \times 100 \times 1 \times 1.08 \times 1.73}{83{,}690} = 2.93 \text{ volts (1.3 per cent)}$$

$$K_2 \left\{ \begin{matrix} \text{6-in. spacing} \\ \text{pf, 90 per cent} \\ \text{No. 5} \end{matrix} \right\} \text{From Fig. 23–14} \qquad K_2 = 1.08$$

Feeder:

Full-Load Current	Starting Current	PF Per Cent	Wire	Fuse	*VD*
183	213	89	No. 00	500	5.9

$$VD = \frac{11 \times 183 \times 200 \times 1 \times 1.13 \times 1.73}{133{,}100} = 5.9 \text{ volts (2.7 per cent)}$$

$$K_2 \left\{ \begin{matrix} \text{6-in. spacing} \\ \text{pf, 89 per cent} \\ \text{No. 00} \end{matrix} \right\} \text{From Fig. 23–14} \qquad K_2 = 1.13$$

The voltage drop to Motor A will be 11.4 volts.
The voltage drop to Motor B will be 8.8 volts.

The allowable drop for motor service is 10 per cent or 22 volts on the 220-volt system. The system is both safe and adequate.

Example l. Design a combined lighting and power feeder to carry the following load.

60 lighting branches of 10- to 100-watt lamps:

Branch A: 25-hp, 230-volt, d-c motor
Branch B: 50-hp, 230-volt, d-c motor

The feeder is to be a 3-wire, Type RP rubber-covered copper installation. Determine the voltage drop, wire size, fuse size, and conduit size for the installation. Feeder 200 ft long.

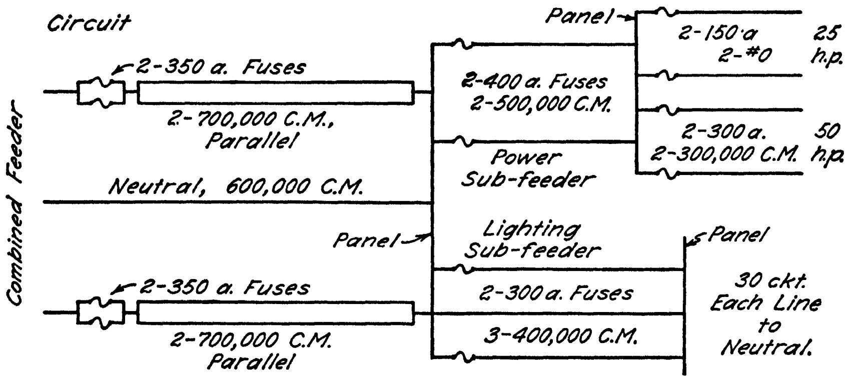


FIG. 27–14. Circuit for Example *l*. Neutral could be reduced under National Electrical Code.

Load	Amount	I_R	I_S	Wire	Fuse
Motors	25 hp	92		No. 0	150 amp
	50 hp	180	225	300,000 *CM*	300 amp
Motor feeder		272	317	500,000 *CM*	400 amp
Lighting subfeeder			261	400,000 *CM*	300 amp
Lighting feeder	60–1000 ckt	261	$\frac{261}{0.8} = 326$	600,000 *CM*	
Combined feeder		533	$\frac{578}{0.8}$ 362	2–700,000 *CM* parallel	2–400 amp parallel

Five conductors in conduit use 80 per cent of wire capacity (Table V–14*A*). Using fuses and wires in parallel, though permitted, is not good practice. It is better to separate the power and the lighting load.

Voltage drop of combined feeder:

$$VD = \frac{11 \times 533 \times 200 \times 1 \times 1 \times 1}{2 \times 700{,}000} = 0.84 \text{ volts to neutral or } 0.76 \text{ per cent}$$

Conduit (Tables VI-14*A* through VI-14*K*):

25-hp branch 1½ in.
50-hp branch 2½ in.
Lighting branches *various*
Motor feeder 3 in.
Lighting feeder 3 in.

Combination feeder:
Table VI-14*E*, 5 conductors, 40 per cent of conduit to be used for wire
Table VI-14*F*, 0.99 sq in. for 700,000 *CM*, 1.3 sq in. for 600,000 *CM*
Total area $(4 \times 1.3 + 1.16)/0.40$, 15.9 sq in. conduit area
Table VI-14*K*, 4½-in. conduit has 15.95-sq-in. area

PROBLEMS

1-14. Determine the inductive reactance per mile for a 3-phase line as the spacing is varied. Use No. 0000 copper wire with triangular spacing. Compute values for 1-, 2-, 4-, 8-, 16- and 32-ft spacing. Draw a curve showing the relationship of X_L versus spacing.

2-14. Compute the capacitive reactance per mile for the line of Prob. 1. Draw a curve showing the relationship of X_C versus spacing.

3-14. The Boulder Dam-Los Angeles transmission line is 266 miles long and has the following line constants per mile: resistance per phase is 0.112 ohm; inductance per phase is 0.00212 henry; the capacity per phase is 0.0136 microfarad. The frequency of the system is 60 cycles. Determine the equivalent π and T circuit for this line.

4-14. If the input voltage (line to line in Prob. 3-14) is 287,000 volts what is the charging current for the line? Use the T circuit for your solution.

5-14. A plant requires 100 kw at 0.8 power-factor lag from a 3-phase source. This power is supplied from a substation 5 miles distant over lines having a resistance of 1.4 ohms and a reactance of 0.75 ohm per conductor per mile. If the line voltage at the load is 6900 volts, what is the line voltage at the substation?

6-14. Design the safe branch circuit to supply a 50-hp, 220-volt, 3-phase, 60-cycle autotransformer starting induction motor to operate over 200 ft of type R copper wire in conduit. Determine the wire size, fuse size, the required size conduit, and the voltage drop.

7-14. Design the safe lighting branch circuit to supply three 1000-watt lamps from a 110-volt, 60-cycle a-c line. The branch is 200 ft long and constructed with type RP copper wire on 4-in. centers. Determine the wire size, fuse size, and voltage drop.

8-14. Design an adequate branch circuit to supply a 10-hp, 230-volt, d-c motor supplied over 300 ft of type R copper wire on 4-in. centers. Determine (*a*) the adequate wire for a 2 per cent voltage drop, (*b*) the correct fuse, (*c*) the voltage drop. Is the wire also safe?

9-14 Design an adequate (2 per cent drop) branch circuit to supply a 10-hp, 220-volt, 3-phase, 60-cycle, resistance starting squirrel cage induction motor over 300 ft of type R copper wire on 4-in. centers. Determine (*a*) the adequate wire, (*b*) the correct fuse, (*c*) the voltage drop. Is the wire also safe?

10-14 Design an adequate (2 per cent drop) branch circuit to supply a 15-hp, 230-volt, d-c motor supplied through a 250-ft circuit of type R copper wire on 4-in. spacing. Determine (*a*) the adequate wire, (*b*) correct fuse, (*c*) the voltage drop. Is the wire also safe?

11–14. Design an adequate (2 per cent drop) branch circuit to supply a 25-hp, 220-volt, 3-phase, 60-cycle, autotransformer synchronous motor operating at a 90 per cent power factor over 250 ft of type R copper wire on 4-in. centers. Determine (*a*) the adequate wire, (*b*) correct size of fuse, (*c*) the voltage drop. Is the wire safe?

12–14. A 3-phase, 220-volt, 60-cycle feeder over a run of 300 ft of type RP wire on 4-in. spacing supplies three motors:

(1) 50-hp, autotransformer starting induction motor
(2) 50-hp, resistance starting induction motor
(3) 50-hp, autotransformer starting synchronous motor operating at 85 per cent power factor.

Determine (*a*) the safe wire size, (*b*) proper fuse, (*c*) voltage drop.

13–14. A 3-wire, 110/220 volt, 1-phase, 60-cycle feeder supplies two branch circuits:

(1) 10-hp, 220-volt, 1-phase, 60-cycle, line starting induction motor, over 100 ft of type RP copper wire spaced on 4-in. centers:
(2) 30–1000-watt, 110-volt incandescent lamps over 125 ft of type RP copper wire on 4-in. centers.

The feeder is also type RP copper wire on 4-in. centers and has a run of 250 ft. Determine for each part (*a*) the proper wire size for safety, (*b*) the proper fuse, (*c*) the voltage drop. What is the voltage drop to the motor to the incandescent lamps?

14–14. A machine is rated at 100 kva, 2300 volts, 3-phase and is wye-connected. The machine has a 2 per cent resistance drop and a 5 per cent reactance drop per phase. Determine the actual resistance and reactance per phase.

15–14. A supply line with 20 ohms resistance, and 55 ohms reactance supplies a rated load of 50 kva at 2300 volts. Determine the resistance and reactance drop of the system.

CHAPTER 15

ELECTRONICS

"Electricity is a physical agent pervading the atomic structure of matter and characterized by being separable, by the expenditure of energy, into two components designated as positive and negative electricity, in which state the electricity possesses recoverable energy." * The advancement in the manufacture and use of electron tubes during the past two decades can be attributed to many important discoveries in the field of atomic and molecular structure. In order to understand the theory and operating characteristics of electron tubes, it is necessary to have some knowledge of the atomic and molecular structure of matter and of the different methods of ionization and radiation.

1. Electrons. According to the most modern theory, the structure of the atom consists of a large positive charge (called the nucleus) which is surrounded by one or more negative charges of small mass (called electrons), which revolve about the nucleus. The nucleus is believed to consist of a closely assembled group of positive and uncharged particles. The net charge of the nucleus is positive and its mass is greater than the electron. Electrons are held within the atom by the electrostatic forces between them and the nucleus. The electron is a unit of electricity (negative in value) which may be considered to be an indivisible material particle. Experimental tests indicate that the electron has the following dimensions:

Charge	1.6×10^{-19} coulombs
	4.8×10^{-10} electrostatic units (esu)
Diameter	2×10^{-13} centimeters (approx.)
Mass	9.1×10^{-28} grams

This mass, when compared with the hydrogen atom (1.66×10^{-24} gram) is equal to 5.42×10^{-4}, or 1/1840 of the hydrogen atom.

A certain number of electrons which are not rigidly bound in the atom exist in all materials. These *free electrons* are able to travel around in the material from one atom to another. Non-conductors have the electrons very closely linked to the nucleus in each atom, with few free electrons; therefore no appreciable current will flow. In good conductors electron movement is easy because many free electrons are loosely

* American Standard Definitions of Electrical Terms—1941.

bound to the nucleus. Although the electrons are loosely bound to a particular nucleus in a metallic atom, no force is present tending to move the electrons in any given direction.

The surface of a metal is a boundary surface for the free electrons and electrons moving at ordinary velocities cannot enter or leave the surface. If an electron is to escape from the surface of a metal, it must have an increased component of velocity acting perpendicularly to the surface, which component is sufficient to overcome its own velocity and stored energy. The amount of energy which an electron requires to free itself from the surface of a metal is called its electron affinity or work function. This energy can be expressed as

$$W = q_0 E_0 \text{ ergs}$$

where q_0 is the charge in statcoulombs, and E_0 is the increase in voltage required to free the electron expressed in statvolts.

If the practical system for voltage is used,

$$W = \frac{q_0 E}{300} \text{ ergs}$$

and

$$E = \frac{300W}{q_0} \text{ volts}$$

This relationship expresses the work function in electron volts. The energy equivalent of 1 electron volt is 1.6×10^{-12} ergs or 1.6×10^{-19} joules. The work function of various metals varies from 1 to 7 electron volts. Table I–15 gives the work function of several metals in electron volts.

TABLE I–15

Metal	Work Function Electron Volts	Metal	Work Function Electron Volts
Barium	2.0	Molybdenum	4.3
Barium-strontium		Nickel	5.0
oxides (BaO, SrO)	1.04	Platinum	6.2
Calcium	2.5	Tantalum	4.1
Carbon	4.5	Thorium	3.0
Copper	4.0	Tungsten	4.5
Magnesium	2.7	Thoriated tungsten	2.6
Mercury	4.4		

The velocity of the electron, in order to leave the surface of a metal for which the work function is approximately 4 volts, must be of the order of magnitude of 10^8 centimeters per second. The emission velocity at

which electrons leave the surface is difficult to attain for some metals and, as a result, they are not used as electron sources. Tantalum, tungsten, thoriated tungsten, thorium, barium, and barium-strontium oxides are some of the electron sources most commonly used commercially.

The three most important methods by which electrons are expelled from the surface of metals are: (1) radioactive disintegration, (2) thermionic emission, and (3) photoelectric emission. The last two are important to the electrical engineer, since the vacuum tube and photoelectric cell are possible only because of these phenomena. Thermionic emission results from heating the metal, supplying the necessary energy to release electrons, and also increasing the thermal energy of the unbound electrons. Photoelectric emission results when the energy of the light falling on the metal surface is transferred to the unbound electrons of the metal.

2. Control of Electron Flow. As mentioned previously, the velocity of the free electrons in the metal can be increased by increasing the temperature of the metal. At ordinary temperatures, electron emission is slight because the average velocity of the electrons within the metal is low. An increase in the temperature of the metal increases the velocity of the electrons and, as they reach the minimum emission velocity, they are emitted from the surface of the metal. If the metal is in an evacuated space, the liberated electrons will gradually fill the space surrounding the electron emitter. These electrons moving at different velocities will collide with one another and, as a result, some will have an increased velocity and others a decreased velocity. The greater portion of the electrons will remain near the surface of the metal though some will move away into space. When the effect of the electron distribution in the space surrounding the emitter is such as to cause as many electrons to reenter as to leave the metal, equilibrium is attained, and the space is fully charged. O. W. Richardson (1901) demonstrated that the electron emission per unit area per unit time in a vacuum depends upon the kind of material and is an exponential function of the temperature. This is expressed by the equation

$$I_s = AT^2\epsilon^{-\omega/kt} \quad \text{amperes per square centimeter}$$

where I_s = current per square centimeter
T = temperature (Kelvin)
ω = work function
ϵ = natural logarithmic base
k = Boltzmann's constant (8.63×10^{-5} volts per degree)
A = constant—see Table I–15

For Richardson's Law to hold, it is necessary that the electrons be removed from the space surrounding the metal as rapidly as they are emitted. The conventional direction of current flow is opposite to the direction of motion of electrons.

An examination of Richardson's Law reveals two important conditions which must be satisfied for a practical thermionic emitter, namely, high operating temperature and low work function. The various metals and metallic compounds which have been mentioned previously consist of those which have a high work function and operate at high temperatures as well as those which have a lower work function and operate at lower temperatures. In the first group are the pure metals, molybdenum, tantalum, and tungsten; in the second group will be found the metallic oxides of barium and strontium.

If a metallic plate is placed in the vicinity of the electron source, some electrons will reach it and the plate will assume the potential of that space (negative with respect to the electron source). When a potential difference is applied between the plate and the emitting source so that the plate or anode has a positive potential with respect to the source, the electrons will be attracted to the plate and a current will flow. The current flow is from anode to cathode. The plate current is also a function of the potential difference for a constant emitter temperature. This relationship, as first stated by C. C. Childs (1911), is expressed as

$$I_p = KE_p^{3/2}$$

where I_p = plate current in amperes
E_p = difference of potential between plate or anode and emitter or cathode
K = constant (depending on dimensions and relative positions of filament and plate).

The relationship depends upon several assumptions which should be mentioned. These assumptions are: (1) there is no gas in the tube to cause retardation of the electrons; (2) the electrodes are parallel plates (infinite); (3) space charge only limits the current flow; (4) electrons do not have an initial velocity; and (5) contact potential at the plate is negligible. It is obvious that all these assumptions cannot be realized and as a result the exponent may differ somewhat from $3/2$ in practice.

In the preceding paragraphs it was shown that the electrons which escape from the surface of the electron source (cathode) will remain in the space surrounding the source. The presence of these electrons in this region will produce an electric field which in turn will tend to decelerate the movement of electrons from the source. The potential in the vicinity of the electron source will decrease to a value which is nega-

tive relative to the source potential. The charge in the surrounding region of the electron source, caused by these free electrons, is called space charge.

If another conductor or plate (anode) which is positively charged is placed in the vicinity of the electron source, its field will attract the electrons, reducing the effect of space charge. As the voltage of this anode is increased, more and more electrons will be attracted from the space surrounding the cathode. There will be a critical value for the anode voltage beyond which only a slight increase in electron flow will occur. In a like manner, the cathode temperature may be increased.

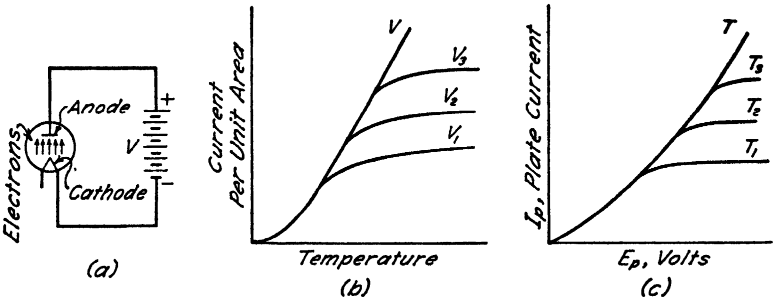


FIG. 1–15. Characteristics of a diode: (a) circuit diagram; (b) relationship as expressed by Richardson's Law; (c) relationship as expressed by Childs' Law.

This will cause greater emission of electrons but which, in turn, will increase the space charge effect. This effect limits the electrons which will reach the anode even though the temperature may be increased further. Figure 1–15 shows the characteristic curves for a two-element or diode tube.

3. Kinds of Electron Tubes. The various tubes may be classified according to: (1) number of electrodes or elements in the tube envelope or (2) type of cathode. For the first classification, the tubes are called diodes, triodes, tetrodes, or pentodes. For the second classification, the tubes are called hot cathode (heated), cold cathode, photoelectric, or mercury pool.

Occasionally, some inert gas is used in obtaining desired operating characteristics for a tube. The ordinary vacuum tube has very desirable characteristics for certain applications such as amplification and rectification. The vacuum tube has a comparatively large loss because the cathode must be heated, a high anode voltage is required, and the tube has a high plate resistance. These may be reduced materially if an inert gas or mercury is introduced into the tube envelope. This will change the operating characteristics of the tube.

4. Hot Cathode Thermionic Tubes. *a. Diode.* The diode or two-element tube is the simplest of the electron tubes. It is used almost entirely with an alternating voltage source. The source voltage applied between the plate and cathode causes the plate to become alternately positive and negative with reference to the cathode. As a result the current flows only when the plate is positive. The tube carries a unidirectional pulsating current during the half-cycle. This rectifying effect, recognized by Fleming in 1905, is the basis of the rectifier tubes used in storage battery chargers. Figure 2–15 shows a characteristic

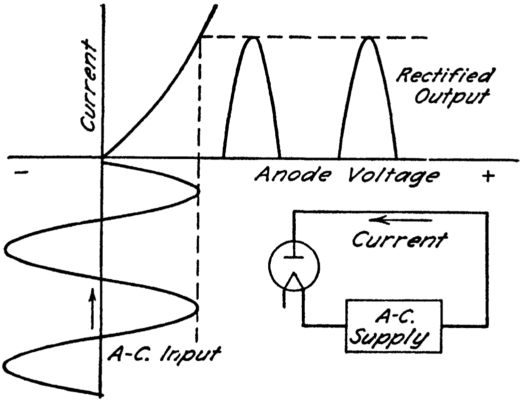


FIG. 2–15. The diode as a rectifier of alternating current.

curve for a diode and rectified current from the tube when alternating voltage is applied. Since the plate voltage-plate current characteristic is not a straight line, the current flow will not be a sine wave for the half-cycle even though the supply is sinusoidal. Figure 3–15 shows the circuit diagrams and rectified waves for half-wave and full-wave rectification. There is a power loss in the tube caused by the tube resistance between plate and cathode and power required to heat the cathode. In some of the large tubes, the plates must be water cooled. This method of rectification is commonly used in radio sets and in high voltage d-c supplies for X-ray work, insulation testing, and corona studies. D-c voltages of 150,000 volts and above are obtained by this method. It is quite common to place the two plates in the same envelope with one cathode to obtain full-wave rectification if the voltage is a few hundred volts.

For half-wave rectifiers for charging storage batteries, the tube may be filled with some inert gas (usually argon). The gas improves the tube performance for low voltages and makes it possible to use it as a battery charger. The "Tungar" (General Electric Company) and "Rectigon" (Westinghouse Electric Corporation) battery chargers use

this type of rectifying tube. These chargers have an output voltage up to approximately 100 volts and a maximum current of around 100 amperes, but usually they are rated at 12 volts and 10 amperes. The half-wave rectifier has an efficiency from 30 to 60 per cent and the full-wave rectifier has a somewhat higher efficiency.

b. Triode. It has been shown that a positive charge on the plate is necessary in order to attract the electrons emitted by the cathode. It

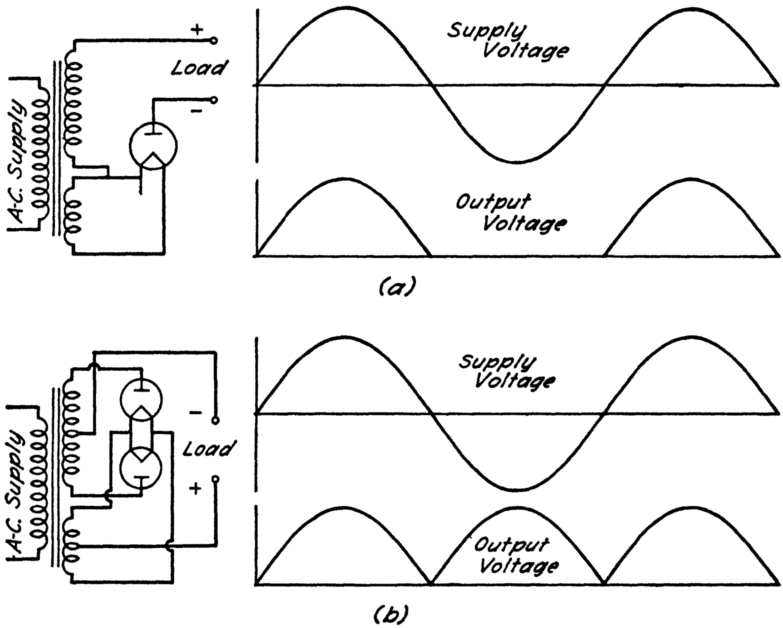


FIG. 3–15. Half-wave and full-wave rectification, using the diode.

has also been shown that, to increase the electron flow, the plate charge must be increased. In order to make the control of electron flow more efficient a third element, the grid, is introduced between the plate and cathode. The grid is of a mesh construction in order to permit the electrons to pass through it. When a triode is used as an amplifier, usually a negative potential is applied to grid relative to the cathode. The number of electrons attracted to the plate will be determined by the combined effects of grid and plate potentials. With the plate at positive potential (the usual condition), making the grid potential more negative will cause a decrease in plate current because the plate is less able to attract the electrons to it. Conversely, as the grid becomes less negative, the plate current increases. Figure 4–15 shows the plate current-plate voltage characteristics of a typical triode as the grid potential is varied.

It can be seen from a study of Fig. 4–15 that a change in grid potential in accordance with some input variation will produce a plate current variation of the same type. Thus the triode becomes an amplifier of the tube input conditions. Figure 5–15 shows a typical circuit using a triode.

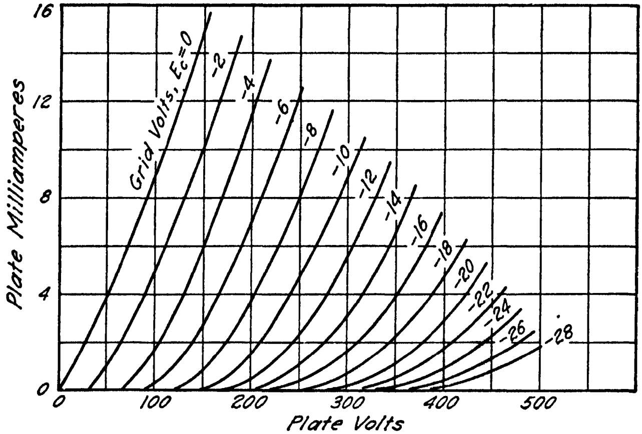


FIG. 4–15. Average plate current-plate voltage characteristics of a typical triode.

The plate, cathode, and grid are, in themselves, part of an electrostatic system and each acts as a plate of a condenser. These capacitances between tube elements are called interelectrode capacitances and, of these, the grid-to-plate capacitance is the most important. This capacitance often acts to produce an electrostatic coupling between the input (grid to cathode) and output (grid to plate) which in turn may make the tube unstable and cause it to perform unsatisfactorily.

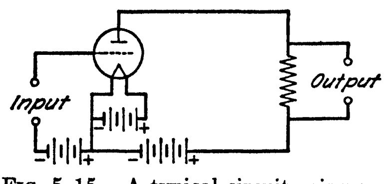


FIG. 5–15. A typical circuit using a triode as an amplifier.

c. Tetrode. It is possible to decrease the grid-to-plate capacitance by adding an additional electrode, called a screen grid, between the plate and grid. The addition of the screen grid makes four elements in the tube, hence the term tetrode. This screen grid is operated at a positive potential with respect to the cathode and at a lower positive potential than the plate. The fact that the screen grid is operated at a positive potential means that electrons will be attracted from the cathode. Since the screen grid has rather large spaces between the wires of the screen most of the electrons which it attracts from the cathode will pass on through it to the plate. In this way the screen grid produces an

electrostatic force which pulls the electrons from the cathode to the plate. In a like manner, it also shields the electrons in the region between the cathode and screen grid so that the electrons near the cathode have very little force exerted on them by the plate. Because of *this action,* as long as the plate potential is higher than the screen grid potential, the plate current is determined more by the screen grid potential than by the plate potential. This condition means that the plate cur-

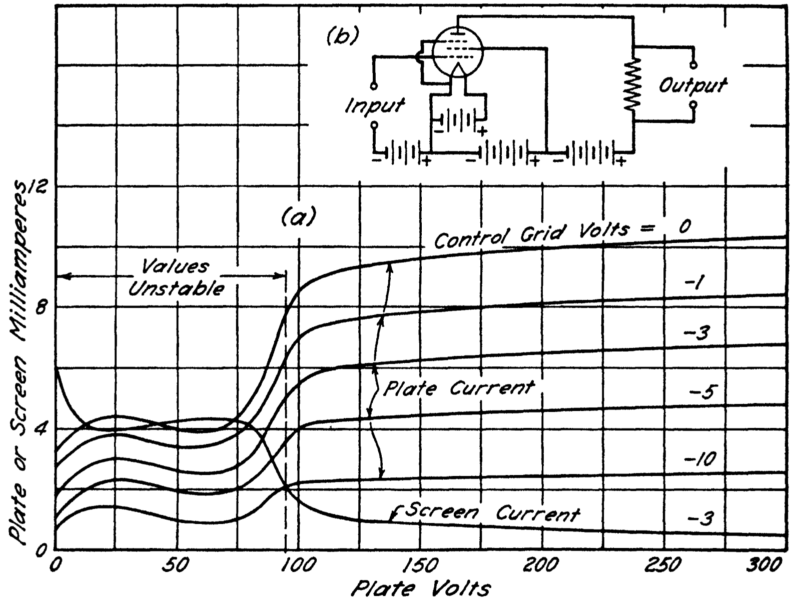


FIG. 6–15. (a) Average plate current-plate voltage characteristics of a typical tetrode. (b) A tetrode circuit diagram.

rent is practically independent of plate potential and as a result much higher amplification can be obtained with a tetrode than with a triode. The tube is more stable at this increased amplification because of lower grid-plate capacitance. Figure 6–15 shows the circuit connection and the plate current-plate voltage characteristic for a typical tetrode.

d. Pentode. It has been assumed in the previous discussion that the conditions of electron flow were such that no undue effects would be experienced as the electrons left the cathode and moved to the plate. The actual conditions are that as the electrons strike the plate other electrons already there may be dislodged. This condition of electron bombardment of the plate by the cathode electrons is called secondary emission. Under ordinary conditions, the electrons knocked from the plate in this manner will return to the plate. In the diode and triode,

very little difficulty is experienced as a result of possible secondary emission. However, in the tetrode, the screen grid is close to the plate and it is of positive potential. This means that some of the electrons will be attracted to the screen grid with the resultant decrease in plate current and permissible plate voltage change in the tube.

In order to correct this condition, a negative potential must be introduced in the region between the plate and screen grid to discourage this movement of bombarded electrons from the plate to the screen grid. This is accomplished by introducing a fifth element between the plate

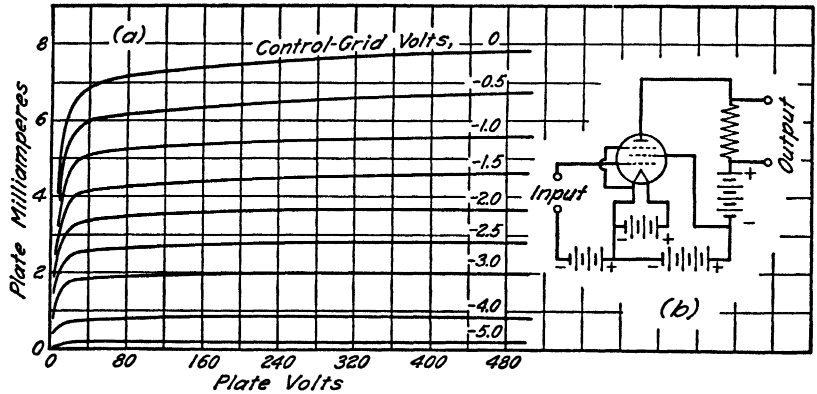


FIG 7–15. (a) Average plate current-plate voltage characteristics of a typical pentode. (b) A pentode circuit diagram.

and screen grid having a potential of the cathode. Because of its negative potential this fifth element or suppressor retards the movement of secondary emission electrons from the plate. This type of tube (pentode) permits increased power output and high voltage amplification. Figure 7–15 shows a typical circuit connection and plate current-plate voltage characteristics for a pentode. The addition of the suppressor in the region of the plate permits an increased plate voltage swing over that of a tetrode with resulting increased performance.

5. Electron Tube Characteristics. The performance and electrical features of electron tubes are usually given for definite conditions and certain relationships. These conditions may be given in tabulated form or in curve form. The relationships in curve form (characteristic curves) may be used not only to determine tube performance but also to assist in calculating additional tube information. The tube characteristics are determined from measurements for the operation of the tube under definite voltages and cathode temperatures in various electrical circuits. The characteristics are usually divided into two classes,

namely, the static characteristics and the dynamic characteristics. The static characteristics are obtained from a study of the tube performance with different values of d-c potentials applied to the tube components. Dynamic characteristics are obtained from different a-c voltages on the control grid of the tube for different values of d-c potentials on the remaining tube components.

The usual static curves consist of two relationships, namely, the plate characteristic and the transfer characteristic. The plate characteristic curves show the plate current as a function of plate voltage for different control-grid bias voltage. The transfer characteristic curves show the plate current as a function of control-grid bias voltage for different values of plate voltage. These curves can be obtained readily from the plate characteristic curves since the same three variables are involved.

The dynamic characteristics include such information as plate resistance, amplification factor, and transconductance. The plate resistance, r_p, of a tube is the reciprocal of the plate conductance. The plate conductance is defined as the quotient of the in-phase component of plate current by the plate alternating voltage, all other tube element voltages remaining constant. Only infinitesimal values are to be used. Thus

$$\text{Plate conductance} \quad g_p = \frac{di_{\text{plate}}}{de_{\text{plate}}}$$

$$\text{Plate resistance} \quad r_p = \frac{1}{g_p} = \frac{de_{\text{plate}}}{di_{\text{plate}}}$$

From this relationship a tube having a plate resistance of 8000 ohms would show a plate voltage variation of 0.8 volt for a plate current variation of 0.1 milliampere.

The amplification factor, or μ (mu), of a tube has been defined as the ratio of the change in plate voltage to a change in control-grid voltage in the opposite direction for a constant plate current and constant voltage for all other tube elements. Again only infinitesimal values are used. From this relationship a tube has a μ of 20 when a plate voltage increase of 2 volts requires a grid voltage decrease of 0.1 volt to maintain the same plate current. The amplification factor can be expressed as

$$\mu = \frac{de_{\text{plate}}}{de_{\text{grid}}}$$

The transconductance, g_m, is a factor obtained by dividing the amplification factor by the plate resistance. A more strict definition of transconductance is the ratio of small plate current change (in amperes) to

the small change in control-grid voltage required to produce the plate current change, with all other tube element voltages remaining constant. For example, a plate current change of 0.1 milliampere is the result of a control-grid voltage change of 0.4 volt and the transconductance is 0.0001/0.4 or 0.0025 mho.

The static and dynamic characteristics are usually given for most tubes in any tube manual. This information is of value in the selection of tubes for a particular use and in amplifier design.

6. Amplification. An electron tube is an amplifier when it is used to reproduce the changes in voltage impressed on the grid circuit as voltage variations across an impedance in the plate circuit. This impedance may be a pure resistance, an inductive reactance circuit, or a transformer. It cannot be a capacitance since the d-c plate current would be blocked by the condenser. Because of their principal applications electron tubes, when used as amplifiers, may be classified as voltage amplifiers, current amplifiers, and power amplifiers.

Tubes for voltage amplification have a relatively high amplification factor and are used where the main object is to obtain a high voltage gain. Usually when a tube is used in this manner, the output voltage is fed into a high impedance such as the grid circuit of another tube.

Tubes for use as current amplifiers are designed to give a large change in plate current for a small grid voltage variation. This means that a tube of this class would have a high transconductance. In addition the tube should be capable of carrying fairly high plate currents.

Tubes designed as power amplifiers will be somewhat of a compromise between the voltage and current amplifiers. This means that the power tube will have a relatively low amplification factor and fairly low plate resistance value. In addition the power amplifier tube should be capable of controlling reasonably high currents at appreciable values of plate potential.

It is desirable that the voltage variation across the impedance in the plate circuit be as nearly an exact reproduction of the grid circuit variation as possible. This condition is not easily obtained and careful study must be given to those conditions where distortion must be a minimum. The operating grid voltage is an important factor in the amount of distortion within the tube. The operation of this amplifying process can be understood better by considering the operation indicated in Fig. 8–15. Figure 8–15 shows the input a-c signal and amplified output superimposed upon the transfer characteristic of the tube. The grid of the tube is given a normal d-c negative bias value on which is superimposed the alternating current voltage of the input signal. As the input variation swings about the normal grid bias, the plate current will vary about

the normal d-c plate current value. A linear grid voltage-plate current characteristic would give a plate current variation exactly proportional to the grid circuit variation.

It can be seen that if the tube is given sufficient negative bias no plate current can flow. This condition is known as the cut-off point. Tube amplifiers are divided into three classes depending upon the normal grid bias used, relative to the cut-off value. The American Standard Definitions (A.S.A.) give the following definitions for the three classes.

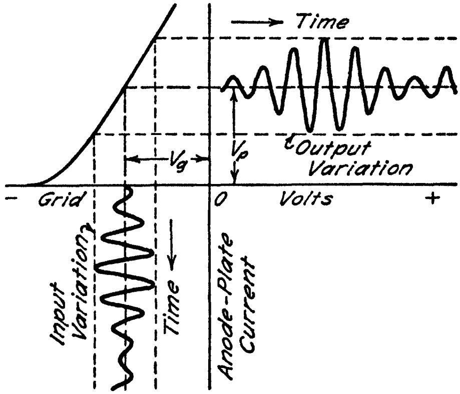


FIG. 8–15. Characteristic tube performance.

(1) Class A Amplifiers. "A class A amplifier is an amplifier in which the grid bias and alternating grid voltages are such that plate current in a specific tube flows at all times."

(2) Class B Amplifiers. "A class B amplifier is an amplifier in which the grid bias is approximately equal to the cut-off value so that the plate current is approximately zero when no exciting grid voltage is applied, and so that the plate current in a specific tube flows for approximately one-half of each cycle when an alternating grid voltage is applied."

(3) Class C Amplifiers. "A class C amplifier is an amplifier in which the grid bias is appreciably greater than the cut-off value so that the plate current in each tube is zero when no alternating grid voltage is applied, and so that plate current flows in a specific tube for appreciably less than one-half of each cycle when an alternating grid voltage is applied."

It was mentioned at the beginning of this article that the impedance in the plate circuit could be resistance, an inductive reactance circuit, or a transformer. The following schematic circuit diagrams indicate how these various impedances can be used to utilize an input signal and transfer it to an output circuit. Figure 9–15 shows the schematic circuit diagram of a two stage resistance-capacitance coupled amplifier. The function of the condensers C is to block any d-c voltages of one tube circuit from appearing in another tube circuit. The grid resistors, R_g, are of the range of 0.5 to 2 megohms and therefore the grid current is very small. The load resistances, R_L, are determined by the amplification desired. The capacitance of the condensers C depends upon the

low frequencies to be amplified. It will be remembered that as the frequency decreases the capacitive reactance increases. This type of amplifier can be used when a reasonably constant amplification over a wide range of frequencies is desired.

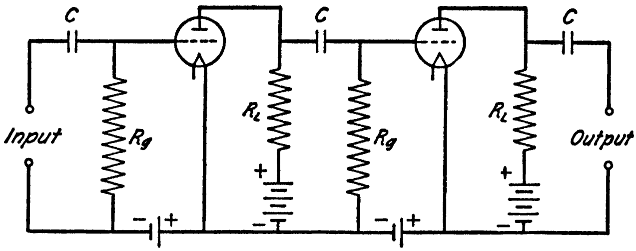


FIG. 9–15. Schematic circuit diagram of a resistance-capacitance coupled amplifier.

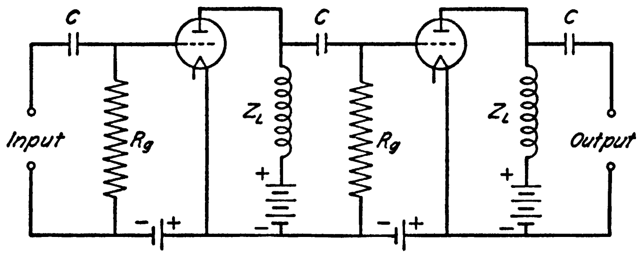


FIG. 10–15. Schematic circuit diagram of an impedance-capacitance coupled amplifier.

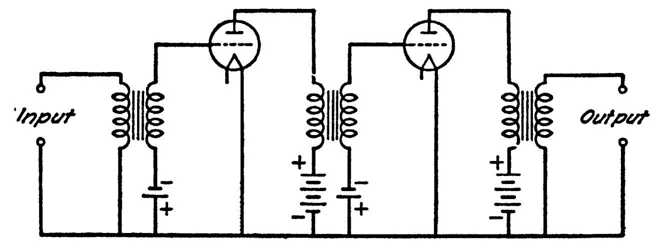


FIG. 11–15. Schematic circuit diagram of a transformer coupled amplifier.

Figure 10–15 shows the schematic circuit diagram of an impedance-capacitance coupled amplifier. Impedances have been used as the plate load instead of the resistances in Fig. 9–15. This type of amplifier is not used extensively in voltage amplifiers although a smaller plate supply can be used since the plate load is chiefly reactance and not resistance.

The tube amplifier circuits also use transformers as a means of coupling the output of one tube to the input of another. By this means

the d-c supply of one tube is isolated from the others and, also, transformer ratios other than 1 to 1 are used to increase the overall amplification above the combined μ's of the tubes involved. The frequency response of a transformer coupled system has been improved in the past few years by the use of special alloy materials for the transformer cores. Figure 11–15 shows a schematic circuit diagram of a transformer coupled amplifier.

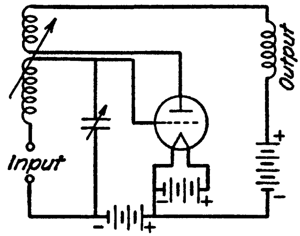


FIG. 12–15. Simple regenerative circuit.

7. Regenerating and Oscillating Circuits. It has been shown that a small variation in the grid supply voltage of a tube will produce an appreciable change in the anode current. The plate supply voltage and the varying current in the circuit result in a comparatively large change in energy output as contrasted to the small energy change which is supplied to the grid circuit. If this output circuit energy is coupled (proper phase displacement considered) with the grid circuit, a part of the output circuit energy can be fed back into the grid circuit and the tube input variation will be increased. This, in turn, will increase the output circuit variation, which will lead to further increase of the input variation. This power feed back is called regeneration, and its effect may be obtained by using a variable coupling inductance coil or a variable condenser, or both. The purpose of these variable devices is to make it possible to adjust the constants of the circuits so that the desired amount of regeneration can be obtained. If the circuits are not tuned to each other, there is no regeneration, because the voltages in the two circuits are not in the proper time relationship. Figure 12–15 shows the circuit diagram of a simple regenerative

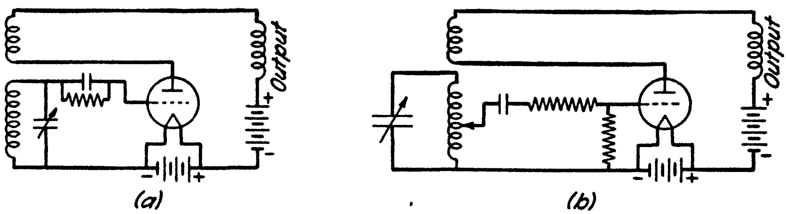


FIG. 13–15.* Oscillator circuits.

circuit. When the regenerative circuit is tuned, the major part of the input impulse is supplied by the plate circuit. This principle is used to a certain extent in radio reception.

If the tube circuit does not receive an external input impulse and the circuit is tuned, the rate of the impulses in the grid circuit will depend

upon the electrical parameters of the circuit. The values of R, $\mathcal{L}$, and C will determine the frequency of the impulses which are fed into the grid circuit by its coupling with the output circuit of the tube. This kind of circuit is called an oscillating circuit. Figure 13–15 shows two oscillator circuits. In order that the circuit may oscillate, the energy fed back from the anode circuit must more than equal the loss in the grid circuit. The oscillator action consists of a definite power surge between the output circuit and the grid control circuit and, by the proper choice of circuit parameters, the frequency of these power surges may be varied over a wide range.

8. Mercury-Arc Rectifiers. The mercury-arc rectifier depends for its operation upon the rectifying properties of a mercury cathode and metallic anodes enclosed in an evacuated chamber. Although the

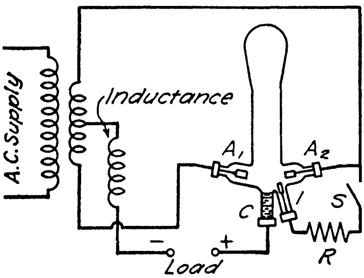


FIG. 14–15. Circuit diagram of a single-phase mercury-arc rectifier.

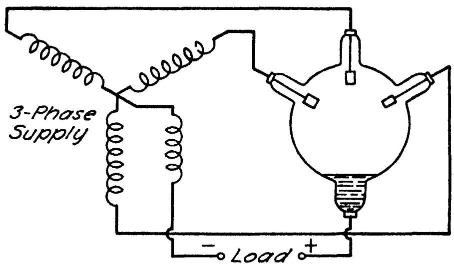


FIG. 15–15. Simplified circuit diagram of a three-phase mercury-arc rectifier.

cathode has a "hot spot" on the mercury pool, this type of rectifier is generally considered as a cold cathode rectifier. The movement of electrons from the arc spot on the cathode to the anode permits the current to pass from the anode to the cathode through the ionized mercury vapor in the tube. The single-phase rectifier has been in use for many years; it consists of a cathode (mercury pool) and two anodes in a glass bulb. Figure 14–15 shows the circuit diagram of a single-phase mercury-arc rectifier. In order to start the rectifier, the tube is tipped slightly and the switch S is closed. The current flows in the starting anode circuit including the load and, as the tube is returned to its operating position, the mercury contact between 1 and C is broken and an arc is formed between the anodes A_1 and A_2 (in succession) and the mercury pool. As the polarity of the a-c wave is reversed, the arc will shift to A_1 or A_2, whichever is positive with respect to the cathode. It is necessary to have some inductive reactance in the load circuit to cause the current to lag behind the voltage. This prevents the arc from dying out when the voltage goes to zero. Figure 15–15 shows a simplified cir-

cuit diagram of a three-phase rectifier which operates on this same principle. Rectifiers, using glass bulbs, are available in capacities of about 50 kilowatts and voltages up to 500 volts. Because these tubes are expensive and are easily broken, their use as rectifiers is limited to small capacities.

Since 1925, the mercury-arc rectifier has been improved by the substitution of metal tanks for glass ones. As this permits water cooling, the rectifier capacity has been greatly increased. This type of rectifier con-

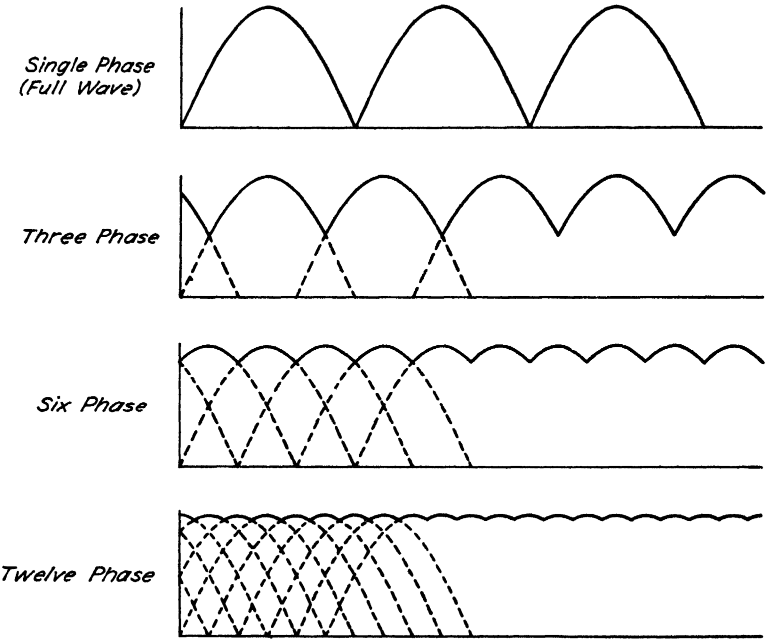


FIG. 16–15. Output voltage waves of a mercury-arc rectifier with different number of phases on the a-c input.

sists essentially of the following: a highly evacuated steel chamber, mercury pool, anodes, a mechanical arc starting anode, a water-cooling system to dissipate the arc losses, and a pumping system to establish and to maintain a high vacuum in the steel chamber. The principle of operation of these rectifiers is the same as that of the single-phase rectifiers, but the capacities are much greater. In the large rectifiers, six- or twelve-phase power is used in order to obtain a smoother d-c output wave. To reduce the anode current, several anodes may be fired at the same time. This multiplicity of anodes is common in heavy-duty rectifiers.

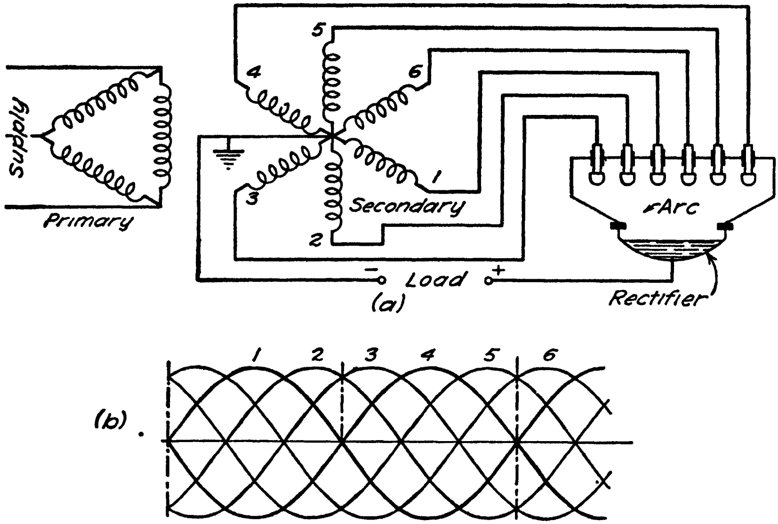


FIG. 17-15. The circuit diagram of a six-phase mercury-arc rectifier and the curves showing the voltage per anode.

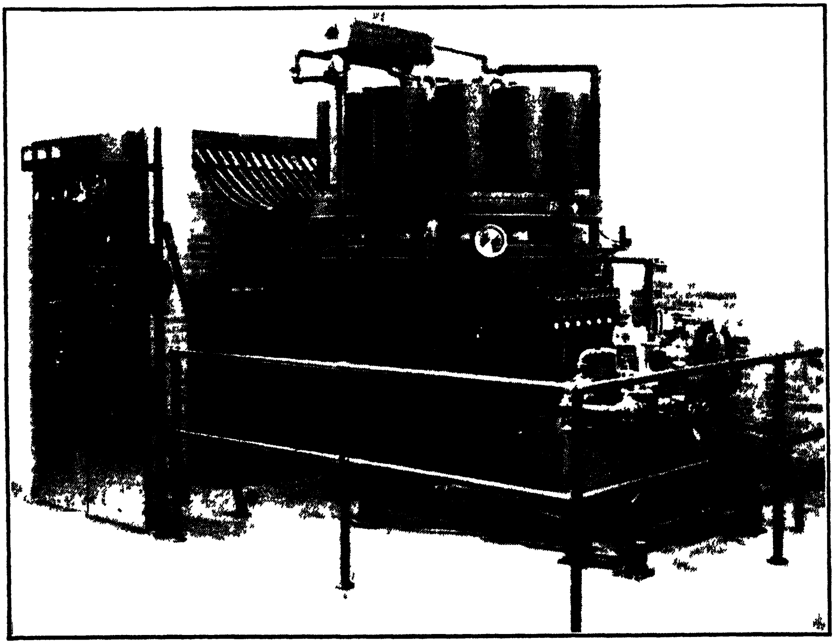


FIG. 18-15. A 1000-kilawatt, 250-volt, grid-controlled mercury-arc rectifier with auxiliaries and control panel. (Courtesy of Allis-Chalmers Mfg. Co.)

Figure 16–15 shows the d-c output for a series of different phase arrangements. The increase in the number of phases shows a more constant d-c voltage from the rectifier. Figure 17–15 shows the schematic circuit for a six-phase rectifier and the voltage wave with the order of firing of the anodes. Figure 18–15 shows the installation of a mercury-arc rectifier.

Rectifiers are rated according to current and voltage, the current capacity being the main factor. The loss in the arc is a function of current and, as a result, the efficiency of the rectifier will increase with

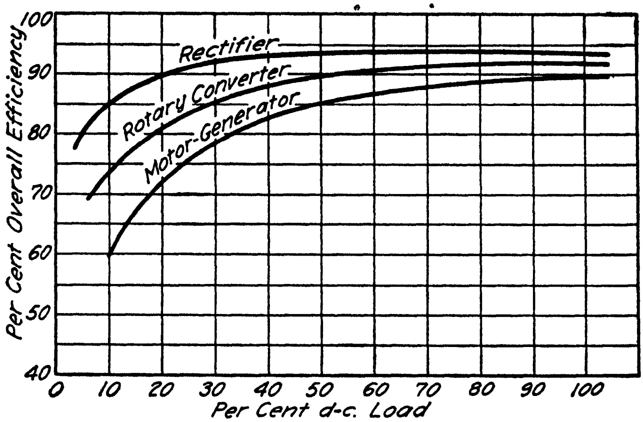


FIG. 19–15. Comparative overall efficiency curves for a 500-kilowatt, 625-volt rectifier, synchronous converter, and synchronous motor-generator set. The a-c supply is 2300 volts at 60 cycles. (Courtesy of Allis-Chalmers Mfg. Co.)

an increase in the d-c voltage rating. The overall efficiency of a rectifier, auxiliaries, and transformers is approximately constant between 20 per cent and full-load rating. The power factor of a rectifier installation varies between 90 and 95 per cent. Figure 19–15 shows the comparative overall efficiency curves for a 500-kilowatt, 625-volt rectifier, synchronous converter, and synchronous motor-generator set.

9. Miscellaneous Rectifiers. The following methods of obtaining rectified currents from an a-c source are used in many installations where only a small amount of power is desired.

a. Mechanical Rectifiers. If a commutator is driven by a synchronous motor and the segments are placed in space phase so that the connections to the d-c circuit are reversed when the a-c supply is reversed, a unidirectional current can be obtained. This device is called a commutator rectifier.

Another mechanical device is the vibrating reed rectifier. Its operation depends upon the vibration of a metallic reed in synchronism with

the a-c supply and making contact to the supply so that a unidirectional current exists in the load circuit. Figure 20–15 shows the schematic circuit diagrams for these two mechanical rectifiers.

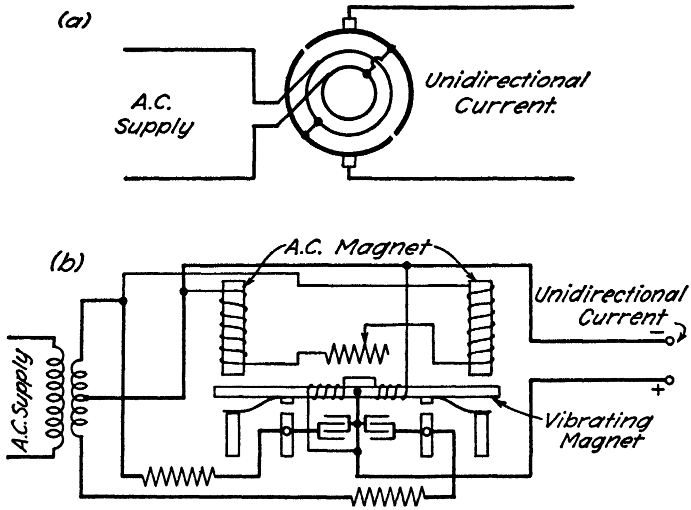


FIG. 20–15. (*a*) Diagram of a rotating commutator mechanical rectifier. (*b*) Simplified circuit diagram of a vibrating mechanical rectifier.

b. Electrolytic Rectifiers. If a lead plate and an aluminum plate are placed in an ammonium phosphate or sodium phosphate solution and connections are made as indicated in Fig. 21–15, a rectified current can be obtained. The current can pass through the solution from the lead to the aluminum plate but cannot pass in the reverse direction. This

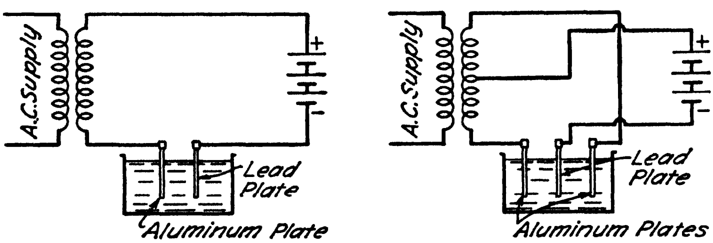


FIG. 21–15. Circuit diagrams for a half-wave and a full-wave electrolytic rectifier.

phenomenon is the result of the formation of an oxide film (aluminum oxide) on the aluminum plate, which is an insulator preventing the current passing from the aluminum plate to the solution.

c. Copper Oxide Rectifier. If an oxidized copper disk (cuprous oxide) and a lead disk are clamped together and an a-c supply is connected in

series with this device and a load, a rectified current will pass through the circuit. The oxide film permits current flow in the direction of copper oxide to copper but opposes the current flow in the opposite direction. By connecting the individual rectifiers in series-parallel, the current capacity of the rectifier installation may become quite large. At the present time, copper oxide rectifiers with capacities up to 2000 amperes are available. The fact that it is a dry rectifier and has no moving parts makes it useful in many installations. The wave shape is similar to that of the ordinary electronic rectifier.

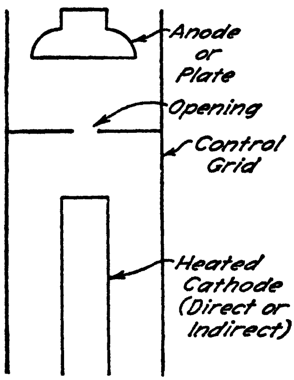


FIG. 22–15. Schematic section of a typical thyratron.

10. Grid-Controlled Gas Tubes. Earlier in this chapter, the use of a diode as a rectifier was discussed. It was shown that, if some inert gas (such as argon or neon) or mercury vapor is introduced into the tube, performance characteristics quite different from those of the vacuum diode are obtained. The introduction of the gas makes the space between the cathode and plate more conductive once the potential has been increased to the point where ionization occurs. The purpose of the gas is to supply the positive ions which are required to neutralize the space charge conditions. Once breakdown occurs, the cathode-to-plate potential becomes low and a large plate current flows.

If a grid is inserted between the cathode and plate, as in the vacuum tube, the tube performance is changed further. The grid in a tube of this kind usually shields completely the cathode from the plate with baffles containing holes for the ionized gas to conduct from cathode to plate. This type of tube is known as a thyratron. Figure 22–15 shows a schematic section of a thyratron with the shielding of both the cathode and plate. Before the gas breaks down, the grid performs in the same way as in the vacuum tube. In the thyratron only a small negative potential on the grid is required to prevent current flow even though the plate potential is high. This condition is the result of the almost complete shielding of the grid. Once the grid voltage becomes more positive than the critical value, the gas is ionized and an arc discharge occurs. This discharge current is not influenced by any grid voltage changes. In other words, the grid has lost control of the plate current flow once the cathode-to-plate path is ionized. The discharge can be extinguished only by decreasing the plate potential below that required to maintain the arc. When the arc has been extinguished, the grid

again gains control of the ionization. The thyratron may be considered a power switch where the control of grid circuit power (fraction

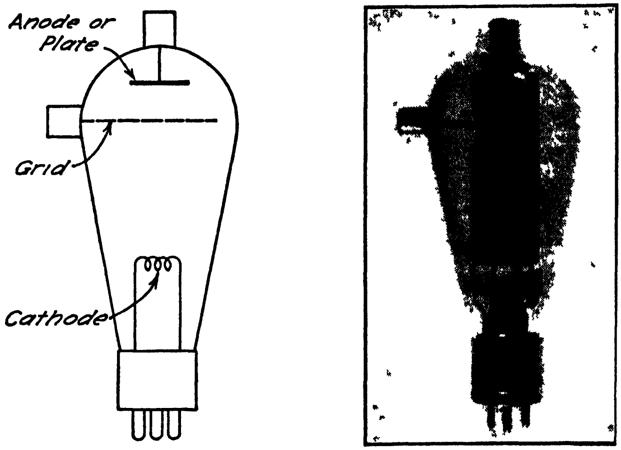


FIG. 23–15 Arrangement of elements in a thyratron.

of a watt) can control kilowatts of power in the plate circuit. Figure 23–15 shows the arrangement of parts in a typical thyratron tube. The cathode is of the directly heated type. The grid is cylindrical and all

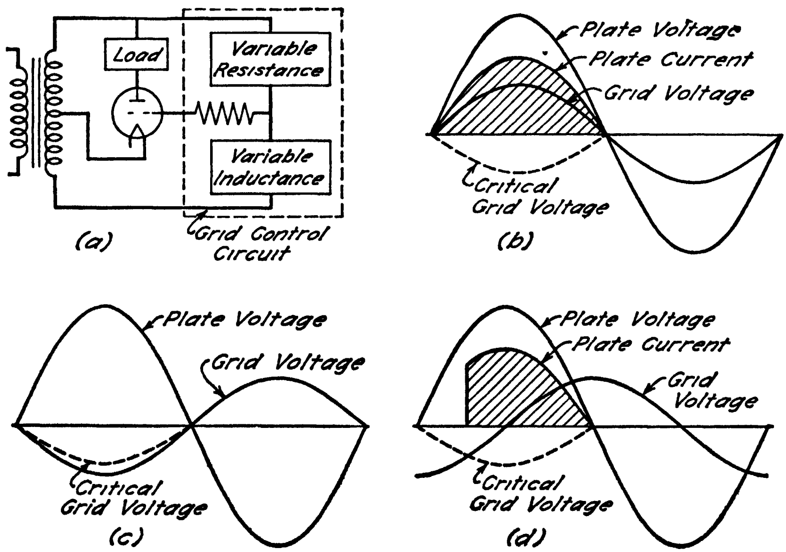


FIG. 24–15 (*a*) Phase control thyratron rectifier circuit (*b*), (*c*), (*d*) Effects of shifting the grid voltage.

the electrons have to pass through the control-grid hole (also cylindrical) as they are attracted to the plate or anode.

The thyratron is commonly used in circuits where alternating voltages are supplied to both the grid and plate circuits. For this type of circuit the phase relationship between the grid and plate voltages will determine the instant when the plate current begins to flow in each cycle. The shifting of the grid voltage in time phase will determine the portion of the current cycle which will flow in the plate circuit. Figure 24–15 shows a typical phase control thyratron rectifier circuit and the effects of shifting the grid voltage. Many variations of this type of circuit are used but, in every case, the important factor is the shifting of the grid voltage in time phase with respect to the plate voltage to control the current flow. The magnitude of the resistance R is influenced by the minimum values of the variable resistance and variable reactance. If these variable factors are sufficient at all times to limit the grid current to a permissible value then the resistance R can be omitted. It will be noted in Fig. 24–15*b*, *c*, and *d* that plate current flows for the portion of the half-cycle after the grid voltage exceeds the critical value for ionization of the gas and arc discharge.

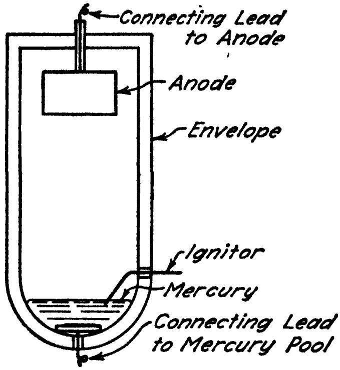


FIG. 25–15. Schematic construction of an ignitron.

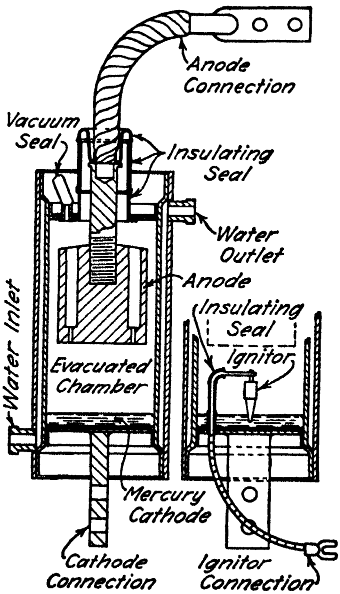


FIG. 26–15. Construction details in an ignitron. (Courtesy of Westinghouse Electric Corp.)

During the past fifteen years another type of three-electrode tube has been developed. The tube is called the ignitron. This name comes from the method used to initiate the discharge within the tube. An electrode called the ignitor is used to start the arc discharge between a cold cathode (mercury pool) and the plate (anode). The ignitor makes contact with the mercury pool at all times and the passing of a current from the ignitor to the mercury causes a small

arc which is followed within a few microseconds by the establishment of the main arc between the anode and cathode. The arc which is established will continue as long as sufficient potential is maintained

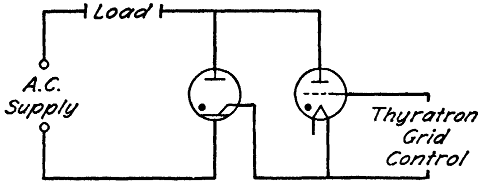


Fig. 27–15. Circuit diagram for a thyratron-controlled ignitron.

in the anode-cathode circuit. When the arc is extinguished because of the lack of potential, an increase in anode potential will not re-establish the arc. The same procedure is used to initiate the arc in

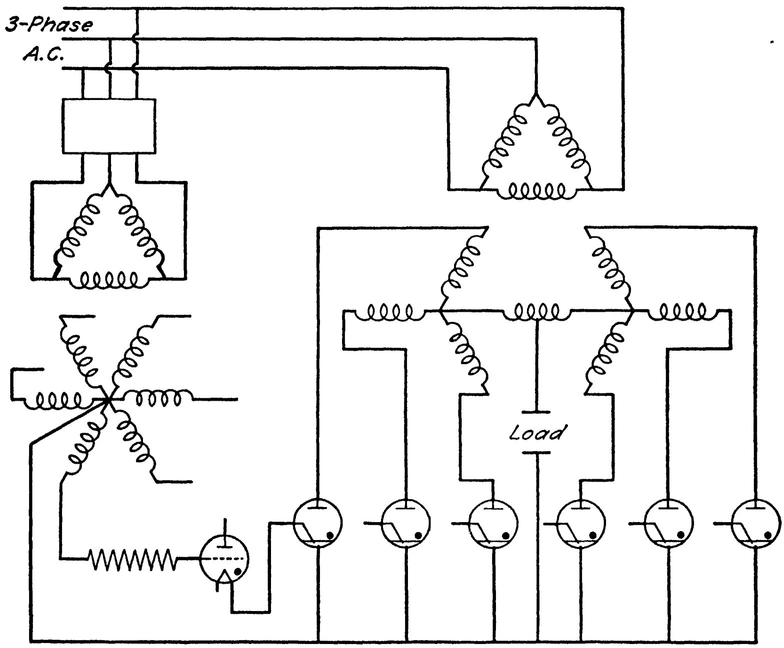


Fig. 28–15. Six-phase ignitron rectifier with thyratron control for the ignitors. Only one ignitor control is indicated.

the ignitron as is used to start the single-phase mercury arc rectifier mentioned earlier in this chapter.

The ignitron has considerably greater capacity than the thyratron and therefore is used more extensively for rectification and control

where large amounts of power are involved. As was shown for the thyratron, a phase-shifting circuit can be used to control the firing of the ignitron. Figure 25–15 shows the schematic construction of an ignitron and Fig. 26–15 shows some of the details of construction. The outer shell is water cooled and this feature enables the unit to control very large amounts of power. Because of the increased current re-

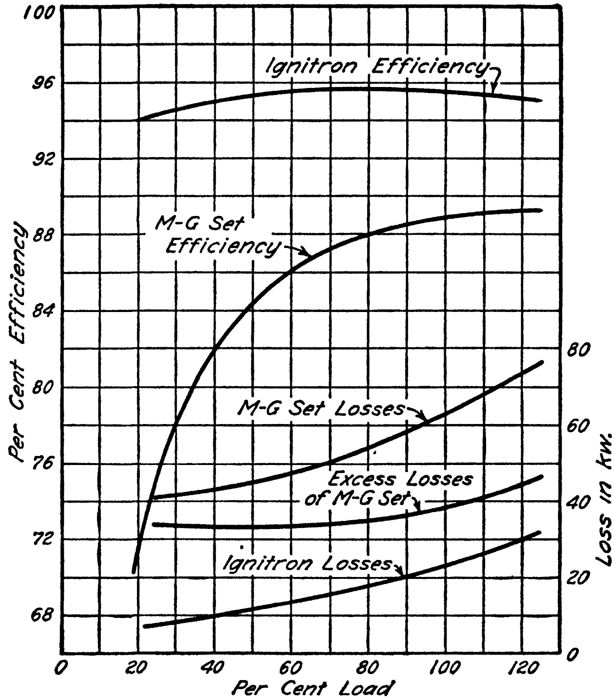


FIG. 29–15. The efficiency and loss curves of a motor-generator unit and an ignitron unit. Each unit is rated at 500 kilowatts, 600 volts, direct current. (Courtesy of Westinghouse Electric Corp.)

quired for the ignitor circuit, it is common practice to use a thyratron to supply this current. By controlling the grid circuit of the thyratron, the ignitron can be controlled. Figure 27–15 shows a simple circuit diagram for a thyratron-controlled ignitron. Figure 28–15 shows the circuit diagram for a six-phase ignitron rectifier with thyratron control on the ignitors. Only one of the thyratron-to-ignitor circuits is shown. It would take six thyratrons for the six ignitors of the system. Figure 29–15 shows the efficiency curves of an ignitron and a motor-generator set. As the voltage and capacity are increased above those indicated, the efficiency of the ignitron unit is increased.

11. Photoelectric Cells. The tubes discussed in the preceding paragraph depend upon a hot cathode for the electron source. It has been found, however, that certain alkali metals will give off electrons at normal temperatures when exposed to light. Cesium, lithium, magnesium, potassium, rubidium, selenium, and sodium are the most sensitive of these metals. The electron emission from the light-sensitive cathode is influenced not only by the amount but also by the color or wave length of the light which strikes the cathode. In most photo-

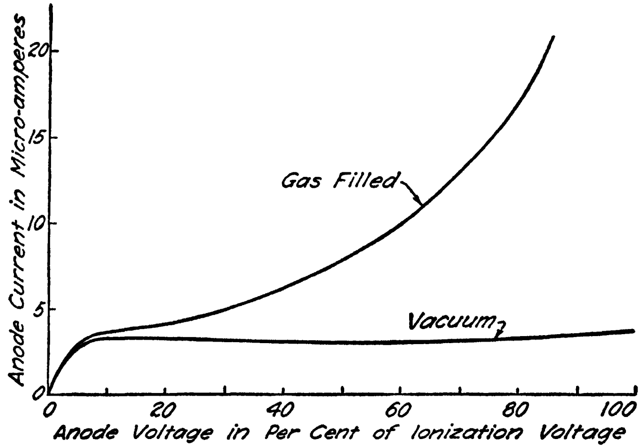


FIG. 30–15. Characteristics of the gas-filled and vacuum photoelectric cells.

electric cells, the greatest effect is produced by the ultraviolet light, with the effect decreasing as the color changes toward the red, or long wave length end of the color spectrum.

Two general types of photoelectric cells are made at the present time; namely, vacuum and gas-filled cells. Vacuum cells are highly evacuated and are used when accuracy and exact proportionality between light flux and electron flow are important. Gas-filled cells are used when larger currents and high sensitivity are required. Figure 30–15 shows the characteristic anode voltage-anode current curves for typical gas-filled and vacuum cells. It will be noted that the anode current continues to increase as the anode voltage is increased. If this voltage approaches a critical value for the tube, the current increases very rapidly and a faint glow appears in the cell. This glow is an indication of ionization and, if allowed to continue, will destroy the light-sensitive surface of the cell. Figure 31–15 shows the effect of different amounts of light on the anode current in a typical gas-filled cell for various anode voltages.

The cathode in the photoelectric cell is cold and, as a result, the electron flow corresponds to only a few microamperes. In order to make use of this small current flow, as influenced by light, the current

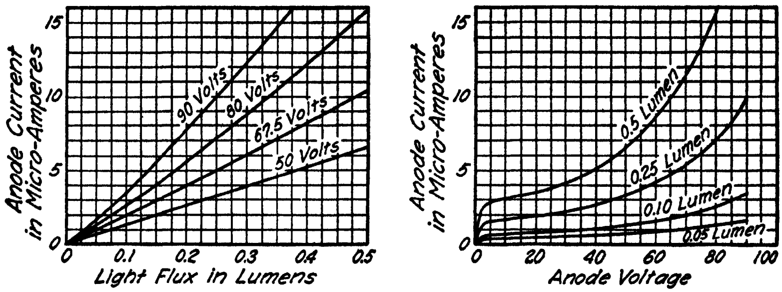


FIG. 31–15. Characteristics of a gas-filled tube.

must be amplified many times. The output of the photoelectric cell may be supplied to the grid circuit of a vacuum tube amplifier and amplified to a value which can be used to operate a sensitive relay or instrument. Figure 32–15 shows a simple circuit using a photoelectric cell and one stage of amplification. The voltage on the photoelectric cell for best operation is usually 75 to 120 volts.

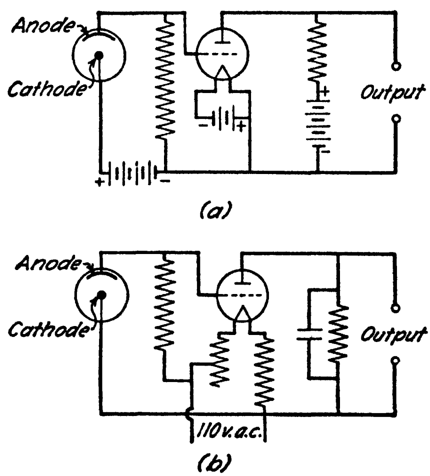


FIG. 32–15. Simple photoelectric cell circuits with one stage of amplification: (a) d-c source; (b) 110-volt a-c source.

The most recent development in photoelectric cells is the oxide-coated, non-evacuated cell. The more common cells of this type use selenium or copper oxide. The cell consists essentially of a thin metal disk on which is impinged a film of light-sensitive material. This metal disk is the positive terminal, and a metal ring in contact with the light-sensitive surface forms the negative terminal. When it is used in ordinary temperatures and light, this type of cell shows no deterioration so that its life seems to be unlimited. The current output of these cells is from 1 to 1.5 microamperes per foot-candle of illumination, uniformly distributed over the sensitive surface, when the cells are connected to a low external resistance. The fact that a low external resistance is

used is a limitation to the useful application of the cell. The low external resistance means no appreciable voltage from the cell and hence the usual electron tube amplifiers cannot be used effectively. The available current from the cell is sufficient to operate low resistance instruments and may be used with low current relays. The fact that no external battery source is required for operation is an advantage; however, the cell is more sluggish in action than the vacuum cell.

12. Industrial Applications. During the past few years the uses of electron tubes in industry have increased very rapidly. The demand for more precise measurements and better control of equipment has caused the development of very complicated yet very sensitive electron equipment. The electrical welding of metals has increased very rapidly in the past decade, and the thyratron and ignitron have been a great aid in obtaining control of the welding arc. The successful development in sound motion pictures can be attributed to both the amplifier tube and the photoelectric cell. The improvement in telephone communication as well as radio has followed the developments in electron tubes.

The following list is given to show some of the industrial applications of the various types of electron tubes.

Humidity control	Lighting control
Viscosity measurement	Smoke control
Voltage relay	Traffic control
Speed control	Automatic sorting
Temperature control	Register control
Noise measurements	Color matching
Telemetering	Power plant control
Train control	Meter calibration
Welding control	Voltage regulation
Sign flashing	Motor speed control

Examples of an electronic voltage regulator, automatic synchronizer, electronic controlled welder, and electronic motor speed-control are discussed in Volume II.

APPENDIX

INDICATING AND INTEGRATING METERS

1. Definitions: *

a. An indicating instrument is an instrument in which the present value of the quantity being measured is indicated by the position of a pointer on a scale. (This assumes correct connections.)

b. A recording instrument (graphic instrument) is an instrument that makes a graphic record of the value of a quantity as a function of time.

c. An electricity meter (meter) is a device that measures and registers the integral of an electrical quantity with respect to time.

2. Indicating Meters. Indicating meters are used to obtain values of current, voltage, power, reactive volt-amperes, frequency, and power factor. The quantities most commonly measured are the current, voltage, and power, and the instruments used for these are the ammeter, voltmeter, and wattmeter, respectively.

The indicating meters of the types for switchboard or portable use must be sturdy in construction and sensitive to variations. It is not desirable to have the meter as sensitive as an oscillograph or a galvanometer element; therefore, the moving element may be increased in weight and strengthened. This increased weight, which increases its inertia, is advantageous because the meter is less sensitive to the small variations in the quantities being measured. The following items are essential to obtain satisfactory performance for indicating meters.

a. To obtain an accurate meter indication, an actuating force proportional to the quantity being measured must be provided.

b. The pointer must come to rest at some fixed place on the scale; therefore, a counterforce or opposition must be provided, the strength of which depends upon the displacement of the pointer from zero. The counterforce increases with the displacement.

c. The instrument must be supplied with some form of damping in order to prevent the needle from swinging back and forth over the scale before finally coming to rest.

d. The scale of the meter should be as uniform as possible in order to aid in reading the indications.

* American Standard Definitions of Electrical Terms, American Institute of Electrical Engineering.

The methods used in obtaining these several requirements lead to the different types of indicating meters. Conforming strictly to one requirement usually leads to a special class of meters and often to rather limited uses for that type of meter. The various types of indicating meters contain the four requirements mentioned, but do not have each requirement developed to the same degree of refinement. Although there are many types of meters manufactured commercially, only the D'Arsonval, Electrodynamometer, and Iron Vane types will be discussed.

3. D'Arsonval Meters. This type of meter is used for all work requiring accurate measurements on d-c systems. In Fig. 1–A is shown

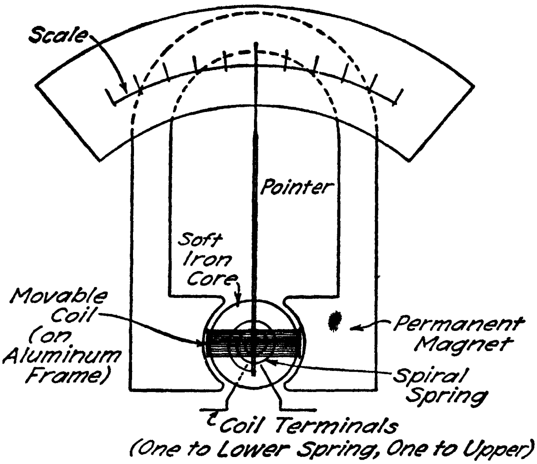


FIG. 1–A. The various parts of the D'Arsonval meter.

the general construction of this type of meter. The operating principle is the same as that of a motor. A conductor, carrying a current, has a force exerted upon it when placed in a magnetic field. If the coil is pivoted so that it is free to move in the magnetic field, although restrained, it will seek a position relative to the magnetic field which is proportional to the current in the moving coil. By using this fundamental principle, the D'Arsonval meter is constructed to give an indication that is proportional to the average value of the current or voltage of a system. For the d-c circuit, the average and effective values of voltage or current are the same. The operating current for this meter is passed through a coil of very fine copper wire wound on an aluminum frame and suspended between the two poles of the permanent magnet. The air gap is made as small as possible and uniform in dimensions. By using a soft iron core in the center of the gap between the two pole faces,

the flux in the air gap of the meter will be of constant strength throughout. This makes it possible for the coil to move in a field of constant flux density and the deflection will depend upon the flux in the air gap and the current in the coil.

The basic principles used in the design of this meter are the same ones which are used for the d-c shunt motor. The torque produced by the current in the pivoted coil, when it is in the field of the permanent magnets, can be expressed as

$$T = KI\phi$$

Since the air-gap flux is constant at all times, the torque will vary directly as the current in the coil and

$$T = K'I$$

The deflection of the coil is proportional to the torque; therefore,

$$\text{Deflection} = K'I$$

The counterforce or opposition is supplied by two spiral springs which also serve as the conductors to the small coil carrying the current. By using these spiral springs, the opposition becomes directly proportional

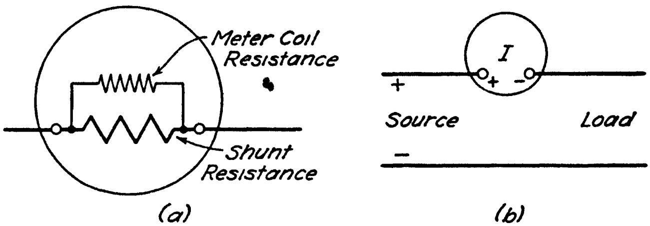


FIG 2–A. (a) Internal circuit diagram of an ammeter with a shunt resistance. (b) Connection of an ammeter in a circuit

to the spring distortion. The meter scale will be uniform throughout its range, because both the applied force and the opposition vary directly with displacement.

The light-weight aluminum frame used for the coil also serves as the damping device for the meter. Any movement of the aluminum frame in the magnetic field produces a generated voltage and eddy currents and, therefore, damping action to the moving meter coil. A few milliamperes of current through the coil will produce a large coil displacement. In order to use this type of meter on circuits to measure large currents or voltages, external resistances are used with the meter and calibrated as a part of the meter.

When the meter is used as an ammeter, it should be used with a shunt resistance as shown in Fig. 2–A. The resistance of the shunt is designed

to match the meter coil resistance and calculated to give the permissible current through the meter coil for the rated current at the meter terminal. Some ammeters have the shunt resistors built into the meter case and the meter is self-contained, but meters measuring currents above 100 amperes usually have the shunt resistors externally connected. Shunts with capacities up to 40,000 amperes are in service.

When the D'Arsonval meter is used as a voltmeter, a high resistance is connected in series with the meter-moving coil to limit the current through it. This series resistance is high compared to the coil resistance in the voltmeter. Figure 3–A shows the circuit connection. The resistance can be tapped at various points, and the full meter scale made

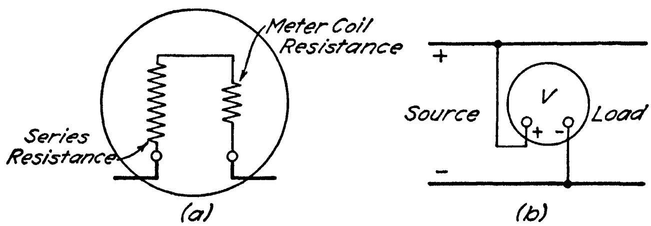


FIG. 3–A. (a) Internal circuit diagram of a voltmeter with a series resistance. (b) Connection of the voltmeter into a circuit.

available for several voltage ranges. By using a tapped resistance, rated current through the moving coil of the meter and, as a result, maximum torque and deflection can be obtained from different voltage sources. This makes it possible to have several voltage ratings for the same meter coil and, by careful selection of these resistances, a multivoltage range meter may be obtained which can measure voltages with a high degree of accuracy and over a wide range. For example, a voltmeter may have a series of voltage ranges such as 0–3, 0–15, 0–150, 0–300, all for one meter element. The resistance of the meter per volt of capacity for the better grade meters is 100 ohms for power circuits and 1000 ohms for meters used in radio circuits.

By the use of properly designed and calibrated shunt and series resistances, the D'Arsonval meter can be used as an ammeter or a voltmeter. The meter is not reversible and will read positive (or up the scale) when the current passes through the meter coil in the proper direction to produce the positive acting torque. All ammeters and voltmeters of this type have the terminals marked positive (+) and negative (−).

4. Electrodynamometer Meters. The previous discussions show that the current and voltage of an a-c system cannot be measured by a

meter that has a permanent magnet producing one of its fields. The torque reverses when the current reverses and the average torque is zero. If, however, the permanent magnet is replaced by a coil producing a reversing field, the meter may be used on a reversing current system. A schematic circuit diagram of this meter is shown in Fig. 4–A. If a pointer is connected to the movable coil, the deflection of the pointer when current passes through the meter will depend upon the fields produced by the stationary and the movable coils. The torque

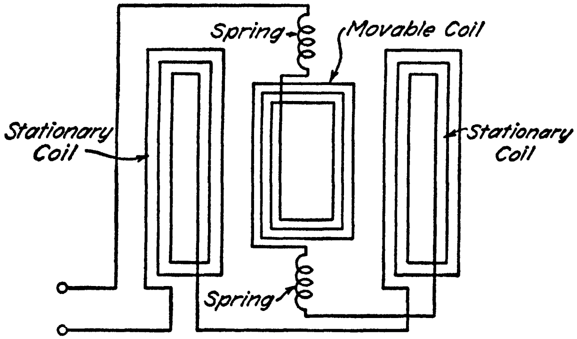


FIG. 4–A. Circuit diagram of an electrodynamometer type of meter. The pointer is fastened to the movable coil and indicates upon a fixed scale.

produced in this type of meter depends upon the strengths of the fields produced by the two separate windings and the sine of the angle between them.

$$T = K\phi_s\phi_m \sin \delta$$

where ϕ_s equals the flux of the stationary coil and ϕ_m equals the flux of the movable coil. The two coils are connected in series (voltmeters and ammeters); the same current passes through the two coils and, since the values of ϕ_s and ϕ_m are proportional to I, the torque varies as $K'I^2$ and the

$$\text{Deflection} = K'I^2$$

The scale for this type of meter will not be uniform as in the D'Arsonval meter, since the deflection varies directly as the square of the current through the meter coils.

When the current reverses in the coils, the torque will not reverse since both fields have reversed and the torque will always be in the same direction. This makes it possible to use the electrodynamometer meter on either d-c or a-c systems. When it is designed to be used on both systems, no iron is used in either coil assemblage as an aid in concentrating the fields since iron produces incorrect indications when used

with a-c systems. Figures 5–A and 6–A show the internal connections for the ammeter and voltmeter. The ammeter is usually constructed to carry a maximum of 5 amperes through the windings, and a parallel

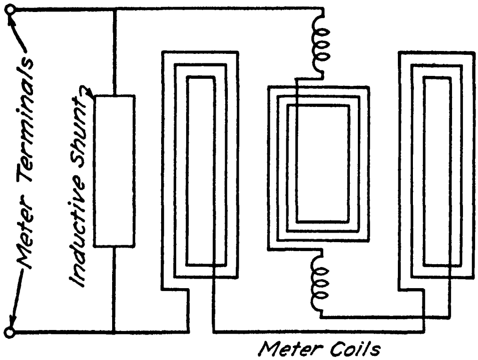


FIG. 5–A. Circuit diagram of an electrodynamometer ammeter and shunt for currents above meter coil capacity. The shunt is usually placed inside the meter case.

impedance having the same ratio of resistance to reactance as the meter coils can be used to limit the meter current.

To measure heavy currents, current transformers are used with the ammeter, and the connections of the current transformer (in the circuit and to the meter) serve the same purpose as the shunt with the D'Arson-

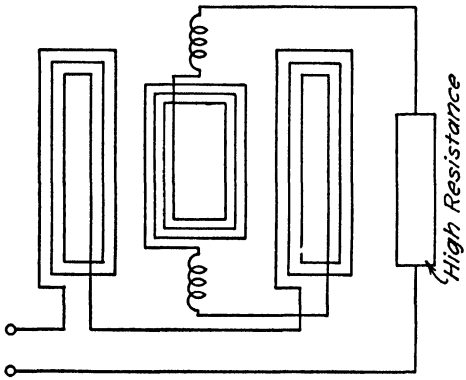


FIG. 6–A. Circuit diagram of an electrodynamometer voltmeter.

val meter. The meter indication multiplied by the transformer ratio gives the true current of the circuit being measured.

In the voltmeter, a series resistance is used to limit the current through the meter. The voltmeter windings are made of fine wire and the meter elements are very sensitive to variations.

A wattmeter using the electrodynamometer principle can be made by using one coil for the current and the other for the potential. The stationary coils are usually the current coils and the movable coil is, therefore, the potential coil. The deflection of the pointer depends upon the current and voltage and the cosine of the angle between them and, therefore, upon the power. When used on an a-c system not having unity power factor the torque varies as the instantaneous power *ei* and the deflection of the meter indicator depends upon the average power.

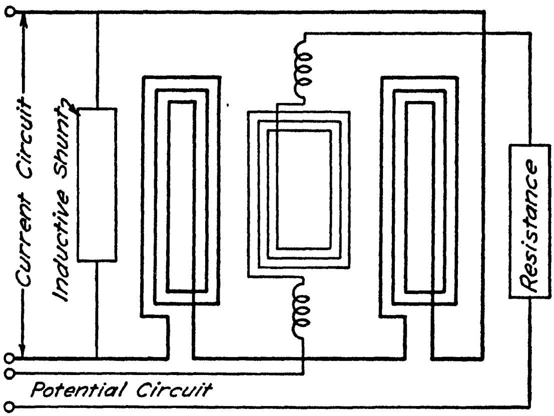


FIG. 7–A. Current and potential circuits of an electrodynamometer type of wattmeter. Each circuit is electrically insulated from the other.

The meter deflection is $KEI \cos \theta$, which is proportional to the average power of the system. Figure 7–A shows the schematic connections of the wattmeter current and potential circuits.

In the wattmeter, the deflection may be reversed, because the coils are separate and the meter may be connected to read negative. In most instances the terminals of the current and potential circuits are marked in order to aid in making the correct connection. The rating of a wattmeter is based upon its current rating and voltage rating, and care must be taken to prevent damage to the meter coils. A wattmeter will give an indication when a current and voltage of the same frequency are used with the meter coils but, in order to have the indication *proportional* to the power of a load, the current and voltage *of the load* must be used. The range of the scale in watts depends upon the current, voltage, and the power-factor limits for which the meter was designed. If the power factor of a load is low, a meter having a low power-factor rating must be used. By changing the angular position of the coil axes with respect to each other, a full-scale deflection of the pointer of the meter

can be obtained for low power-factor loads. This makes it possible to read the low power-factor loads with the same degree of accuracy as the better power-factor loads. The individual meter is rated according to voltage, current, and power factor.

The coils of the meter have their ratings but the coils can be inclined on their axes (with respect to each other) so that a full-scale deflection can be obtained for lower power-factor loads.

5. Iron Vane Meters. The iron vane meter consists of two iron vanes placed in a magnetic field. One of the vanes is stationary and the other is free to rotate on an axis. The vanes are parallel when the field is zero but, with a field present, a repulsion between the two vanes causes the movable vane to rotate on its axis. If a spring is used as the counterforce, the meter can be calibrated and the scale deflection made proportional to either the current or voltage. Figure 8–A shows the meter construction. The vanes are made of soft iron and, as a result, the magnetism can be reversed rapidly in the magnetic circuit with little distortion. This type of meter is applicable to either d-c or a-c circuits and will give a deflection proportional to the square of the current through the coil. The distance between the vanes and their position with respect to the exciting coil of the meter will affect both the torque produced and amount of deflection of the pointer.

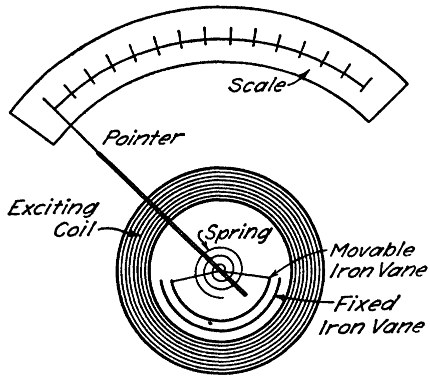


FIG. 8–A. Iron vane meter showing principles of design. The shape of the iron vanes aids in making the scale divisions more uniform.

To aid in giving a more uniform scale, some of the modern meters have the coils at an angle with respect to the stationary field; others will have specially shaped vanes. Either of these design features will cause a deviation from the normal scale deflection for a given current, and it is possible to approach a reasonably uniformly divided scale within the working range of the meter. The iron vane type of meter is very sturdy, but more power is required to operate it. The operating parts of the ammeter and voltmeter are practically the same, the coil for the ammeter consisting of a few turns of large wire, and the voltmeter coil consisting of many turns of fine wire. The voltmeter coil has a resistance of 10 to 15 ohms per volt.

When this type of meter is used on d-c circuits, readings with the current passing through the coil in both directions should be averaged in order to correct for field distortion.

6. Integrating Meters. The integrating meter is used to record the energy consumed during a period of time. This may be represented by the expression

$$\mathscr{E} = \Sigma \int p\,dt \qquad p = \text{instantaneous power}$$

$$e = \text{instantaneous voltage}$$

or

$$\mathscr{E} = \Sigma \int ei\,dt \qquad i = \text{instantaneous current}$$

$$t = \text{time}$$

The counterforce or opposition of the indicating meter is replaced in the integrating meters by a set of permanent magnets with a disk cutting the field of the magnets and producing eddy currents in it. The opposition can be adjusted to be directly proportional to speed and, when the driving force is correctly adjusted to be proportional to the instantaneous power, the speed of rotation of the meter shaft will be directly proportional to the instantaneous power. A counting device to record the revolutions of the shaft will indicate the total energy consumed.

Integrating meters are used to record kilowatt hours, reactive volt-ampere hours, and ampere hours.

7. D-C Watthour Meters. There are two types of d-c watthour meters, namely the commutating type and the mercury type. In the commutating type, the operation is the same as for a commutating motor without any iron in the magnetic circuit. The main-field current of the meter is the current of the load being recorded, and the armature current of the meter motor is proportional to the voltage across this load. The torque producing rotation is caused by the fields, which are, in turn, produced by the current in the armature and the current in the main field. These currents are proportional, respectively, to the load voltage and current; therefore, the torque is proportional to the load voltage and current. Figure 9–A shows the construction of the commutating meter. The compensating coil is used to correct for bearing and gear train friction, which, under light load operation of the meter, is the major counterforce. The mercury meter (Sangamo Electric Company) consists of a slotted disk floating in a pool of mercury. As the current passes through the mercury and copper disk in the magnetic field produced by the potential coils, a torque is produced, causing the copper disk to rotate. As the density of the field is proportional to the

voltage of the load, and the current through the mercury pool is proportional to the load current, the torque produced is proportional to their product, which is the power. The diagram of Fig. 10–A shows the operating principle and schematic diagram of the mercury type-watthour meter. The friction compensation consists of a high resistance thermocouple that can be adjusted to give the proper amount of current through

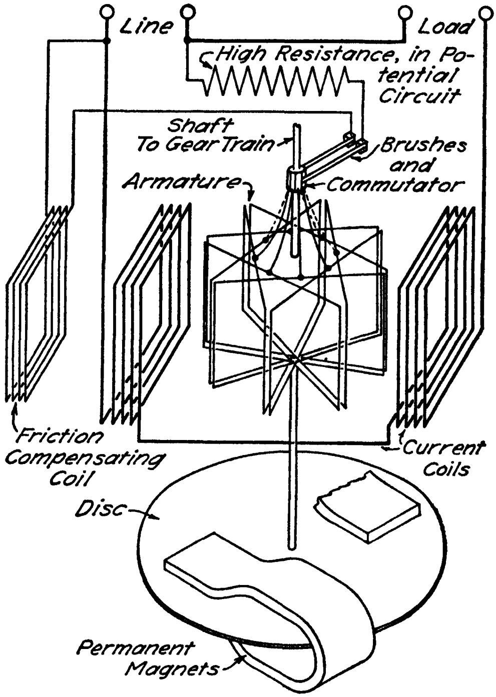


FIG. 9–A. Construction diagram of a d-c commutating watthour meter.

the operating element at all times and, in this way, produce enough torque to balance the effects of friction.

8. A-C Watthour Meters. The a-c meter operates on the induction motor principle and, hence, is usually called the induction watthour meter. As in the d-c meters, two fields are produced: one proportional to the current, the other proportional to the voltage. The retarding force or opposition is produced by the use of a disk in the field of a permanent magnet. This same disk is used to produce the driving torque of the meter. The diagrams of Fig. 11–A show the magnetic and electrical circuits of a single-phase induction watthour meter. The potential coil P, consisting of many turns of fine wire, if connected

across the line and with constant frequency, will carry a current proportional to the line voltage. The current coils CC' are connected in series

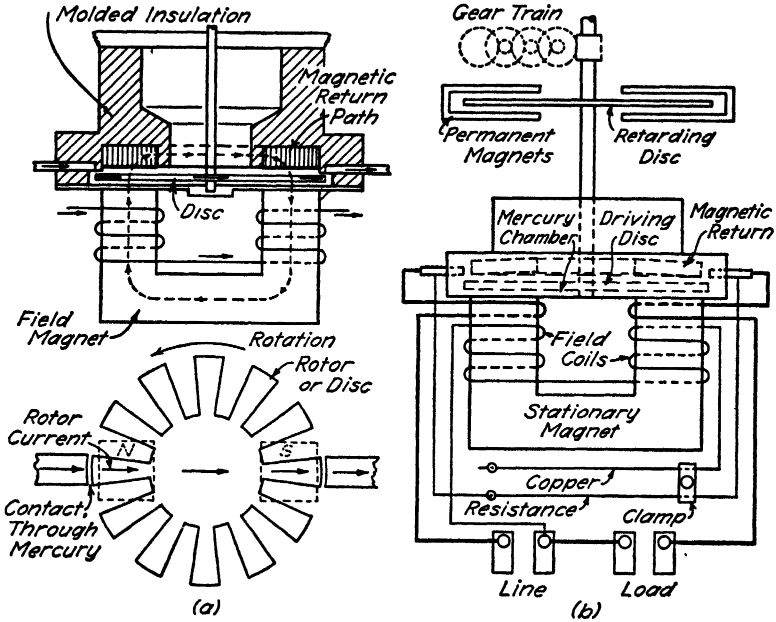


FIG. 10–A. Construction diagrams of the mercury type of watthour meter.

with the load and carry the load current. The current in the potential coil establishes a magnetic field in the laminated iron circuit linked with the coil. At the air gaps the flux spreads and part of it passes through

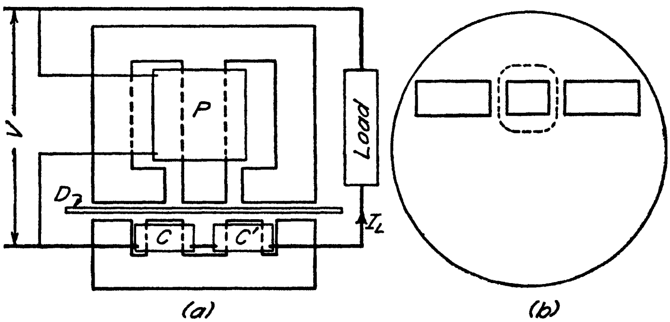


FIG. 11–A. Magnetic and electrical circuits of the induction type of watthour meter.

the disk D and, because the flux is alternating, will cause an electromotive force to be induced in the disk. The electromotive force induced in the disk causes a current to flow in it (Fig. 11–A*b*). This disk current

I_D will lie in the field produced by the coils CC' and there will be, as a consequence, a torque produced tending to rotate the disk. If the field produced by the coils CC' is in time phase with I_D, the average torque produced will be proportional to the product of I_D and the field or, since the field is produced by the load current I_L, to the product of I_D and I_L. If I_L is not in time phase with I_D, the average torque will be proportional to the product of I_D and the component of I_L which is in phase with I_D. Since I_D is proportional to the voltage V, the torque is proportional to the product of V and the component of I_L in phase with V, or to the power of the load circuit. The counterforce or opposition is created by the permanent magnets which produce a countertorque proportional to the speed. The totalizing of the disk revolutions, therefore, will represent the energy consumed.

INDEX